THEORETICAL PHYSICS

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VOL. I. MECHANICS AND HEAT.
NEWTON—CARNOT.

VOL. II. ELECTROMAGNETISM AND OPTICS.
MAXWELL—LORENTZ.

VOL. III. RELATIVITY AND QUANTUM DYNAMICS.
EINSTEIN—PLANCK.

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VOL. I

MECHANICS AND HEAT
NEWTON—CARNOT

WITH EIGHTY DIAGRAMS

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PREFACE

THE purpose of the present work is to present an account of the theoretical side of physics which, without being too elaborate and voluminous, will nevertheless be sufficiently comprehensive to be useful to teachers and students. This, the first volume, deals with mechanics and heat ; the second volume will be devoted to electromagnetism and optics and possibly the introductory part of relativity ; while the remaining volume will deal with relativity and quantum dynamics.

The contents of this part are based on the notes of lectures delivered at one time or another at Bedford College and King's College (London). In selecting the subject-matter I have been influenced chiefly by its importance from the point of view of exhibiting the unity of physical theory and in a secondary degree by any special interest, historical or other, commending it, or by its suitability as a means of preparing the ground for more important things to follow. The unavoidable incompleteness is compensated to some extent by the bibliographical references and notes appended to many of the chapters.

Each part of the subject is developed in a way which follows, broadly speaking, its historical growth, and this first volume is entirely ' classical ', the dynamical part of it being based on the foundations of Newton. At the same time every opportunity that presents itself is utilized to open the way for the description of modern developments of physical theory which will occupy later parts of the work. The methods of elementary vector and tensor calculus are introduced at the outset and consistently followed, partly on account of their fitness and utility, and partly as an introduction to a more complete account of tensor calculus which will have a place in a later volume.

Care has been taken to make the nature of fundamental principles as clear as possible and everything is developed from the simplest beginnings. No very serious demands are made on the mathematical equipment of the reader. A certain acquaintance with the elements of the calculus and analytical

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geometry is assumed and any mathematical methods which extend beyond this are explained as they may be required.

Needless to say, I have derived much assistance from many classical works and papers ; in fact from most of those mentioned in the bibliographical appendices. I am indebted to Dr. Maud O. Saltmarsh for reading the proofs.

W. W.

January, 1931

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THEORETICAL PHYSICS

§ 1. GENERAL INTRODUCTION

PHYSICAL science, in the restricted sense of the term, is concerned with those aspects of natural phenomena that are regarded as fundamental. Broadly speaking it investigates things with which we are brought into immediate contact through the senses, hearing, touch and sight; and it includes, among others, the familiar sub-divisions **sound, heat** and **light.** But *sense perceptions themselves* hardly enter into physics. Indeed, they are deliberately excluded, as far as may be, from physical investigations. Spectra are observed photographically and the *colour* of a spectral line is really not a thing in which the physicist takes any interest. No great effort of imagination is needed to conceive the possibility of photometric devices whereby a completely blind observer might carry out for himself all the observations on a spectrum which have any significance for physics. Temperature is not measured by feeling how warm a thing is, nor in acoustical investigations do we rely on the sense of hearing. In fact, the use of human senses is practically confined, in experimental physics, to the observation of coincidences, such, for example, as that of the top of the mercury column in a thermometer with a mark on the scale of the instrument, or that of the spider line in a telescope with a star or a spectral line, and the associated coincidence which gives the scale reading.

Physical science is the cumulative result of a variety of closely correlated activities which have given us, and are adding to, our knowledge of what we shall call the Physical World, and the present treatise is an attempt to present, in outline, a connected account of the body of doctrine which has grown out of them.

There are three well-defined periods in the development of the theoretical side of physics since the time of Galileo. The earliest of these, which we may call the ' matter and motion ' period, came to an end in 1864 when Clerk Maxwell's electro-

magnetic theory of light appeared.[1] The physicist of this period conceived the world as built up, roughly speaking, of minute particles (atoms) endowed with mass or inertia and capable of exerting forces (gravitational, electric, etc.) on one another. Their behaviour and mutual interactions were subject to certain dynamical principles, summarized in Newton's *laws of motion*. A phenomenon was considered to be satisfactorily accounted for when it could be represented as a mechanical process ; when it could, as it were, be reproduced by mechanical models differing merely in scale from something that might be constructed in a workshop. This mechanical physics was extraordinarily successful, and was tenaciously adhered to and defended, even so recently as the opening years of the present century, as the following quotation from the preface to the first edition of an admirable work on the theory of optics [2] will show.

' Those who believe in the possibility of a mechanical conception of the universe and are not willing to abandon the methods which from the time of Galileo and Newton have uniformly and exclusively led to success, must look with the gravest concern on a growing school of scientific thought which rests content with equations correctly representing numerical relationships between different phenomena, even though no precise meaning can be attached to the symbols used.'

The second period, from 1865 till the opening years of this century, has a transitional character. Maxwell's theory (which, it may be remarked, united the previously disconnected provinces of light and electricity) led eventually to the abandonment of the effort to establish electrical phenomena on the old-fashioned ' matter and motion ' basis and placed ' electricity ' on equal terms by the side of ' matter ' as a building material for the physical world. In the 'eighties indeed the most characteristic property of matter, namely mass or inertia, was successfully accounted for in electrical terms, and attempts began to be made (with some success) to provide a purely electrical basis for theoretical physics.

Distinguishing marks of the present period of theoretical physics (since 1900) are the development of the quantum and relativity theories and the consequent overthrow of the sovereignty of Euclidean geometry and Newtonian dynamics. These latter, however, retain their practical importance in almost undiminished measure, and it would indeed be inaccurate to

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[1] It is noteworthy that Maxwell wrote a little book called *Matter and Motion* which, though he was the inaugurator of a new epoch in physics, presents a very fair picture of, and indicates his sympathy with, the ideals and aims of the earlier period.

[2] Schuster : *An Introduction to the Theory of Optics* (Arnold, 1904).

speak of them as untrue or disproved ; but they now appear as limiting cases of the more comprehensive modern theories. The old problems of the explanation of electrical phenomena in mechanical terms, or of matter in electrical terms, have no longer any significance, and physical theory is approximating more and more to a vast and unified geometrical structure such as was not dreamt of in the philosophy of Euclid or Newton.

FOUNDATIONS OF EUCLIDEAN TENSOR ANALYSIS

§ 2. Scalars, Vectors and Tensors

A PHYSICAL quantity which can be completely specified by a single numerical statement (the unit of measurement having once been chosen) is called a **scalar.** Examples of scalars are electric charge, mass, temperature, energy and so forth. A **vector** is a physical quantity associated with a direction in space. For its complete specification *three* independent numerical statements are necessary. The typical example of a vector is a **displacement.** If a small body or particle is given a series of displacements represented by AB, BC, CD, DE (Fig. 2), which are not necessarily co-planar, it is obvious that these bring about a result equivalent to the single displacement represented by AE. The displacement AE is called the **resultant** of the displacements AB, BC, CD, DE. Any vector can be represented in magnitude and direction by a displacement. Examples of vectors are **force, velocity, momentum, electric field intensity** and so on. If, for instance, four forces are applied to a body (to avoid irrelevant complications we shall suppose them all to be applied at the same point in the body) and if a straight line AB (Fig. 2) be constructed having the direction of the first force and a length numerically equal to it in terms of some convenient unit, and if a second straight line BC be drawn to represent the second force in a similar way and so on ; then the four forces are equivalent to a single or **resultant** force which is represented in magnitude and direction by AE. The three independent numerical data which are necessary to express the vector completely may be given in various ways. We may, for example, give the **absolute** value of the vector, i.e. the length of the line AB or BC (Fig. 2), representing it ; in which case we have to give two additional numerical data to fix its direction

Fig. 2

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relative to whatever frame of reference we may have chosen. The three independent data, however they may be chosen, are called the **components** of the vector. It is usual, however, to restrict the use of the term ' component ' in the way indicated in the following statement : **Any vector can be represented as the resultant of three vectors which are parallel respectively to the X, Y and Z axes of a system of rectangular co-ordinates. These three vectors are called its components in the X, Y and Z directions.** Unless the contrary is stated, or implied by the context, we shall use the term ' component ' in this more restricted sense. If A represents the absolute value of a vector, we shall represent its components by A_x, A_y and A_z and refer to it as the vector **A**, or the vector (A_x, A_y, A_z). The statement ' A_x, A_y and A_z are the X, Y and Z components of the vector **A** ' may conveniently be expressed in the abbreviated form :

$$\mathbf{A} \equiv (A_x, \ A_y, \ A_z).$$

It is clear that when a vector is represented by a line drawn from the origin, O, of a system of rectangular co-ordinates to some point P, its components are the co-ordinates of P.

Besides scalars and vectors we have still more complicated quantities, or sets of quantities, called **tensors.** A tensor of the **second rank** requires for its complete specification 9 or 3^2 independent numerical data, which are not necessarily all different. Just as a vector can be represented by a displacement, so can a tensor of the second rank be represented, in its essential properties, by a *pair* of displacements. This will be more fully explained later. The state of stress in an elastic solid is an example of such a tensor. It has become customary in recent times to use the term ' tensor ' for all these different types of physical quantities. A scalar is a tensor of **zero rank**; it requires for its specification 3^0 or 1 numerical datum. A vector is a tensor of the **first rank,** requiring for its specification 3^1 independent numerical data and so on.

§ 2·1.   Scalar and Vector Products

The **inner** or **scalar product** of two vectors is defined to be a scalar quantity numerically equal to the product of their absolute values and the cosine of the angle between their directions. If the absolute values are **A** and **B,** and if the included angle is θ, the scalar product is **AB** cos θ. It is convenient to abbreviate this expression by writing it in the form (**AB**) or (**BA**). A very important instance of a scalar product is the work done by a force when its point of application is displaced.

If, for example, **A** represents a force (which we shall suppose to be constant), **B** the displacement of the point where it is applied, and θ the angle between their directions, the work done is expressed by **AB** cos θ or briefly by (**AB**).

To elucidate the properties of the scalar product it is convenient to represent the vectors **A** and **B** by displacements from the origin O of rectangular co-ordinates (Fig. 2·1). Let the terminal points, p and q, of the displacements be joined by a straight line, the length of which is represented by t. Then we have

$$t^2 = \mathbf{A}^2 + \mathbf{B}^2 - 2\mathbf{A}\mathbf{B}\cos\theta.$$

Since the co-ordinates of p and q are (A_x, A_y, A_z) and (B_x, B_y, B_z) respectively, it is evident that t is the diagonal of a parallelopiped, the edges of which are parallel to the axes X, Y and Z and equal respectively to $|A_x - B_x|$, $|A_y - B_y|$ and $|A_z - B_z|$, the symbol $|x|$ being used to represent the *absolute* value of x. Therefore

$$t^2 = (A_x - B_x)^2 + (A_y - B_y)^2 + (A_z - B_z)^2.$$

If we remember that

$$\mathbf{A}^2 = A_x{}^2 + A_y{}^2 + A_z{}^2 \quad . \quad . \quad . \quad . \quad \textbf{(2·10)}$$

and

$$\mathbf{B}^2 = B_x{}^2 + B_y{}^2 + B_z{}^2,$$

we find on equating the two expressions for t^2

$$\boxed{\mathbf{A}\mathbf{B}\cos\theta = A_xB_x + A_yB_y + A_zB_z} \quad . \quad . \quad \textbf{(2·11)}$$

This important result may be expressed in words as follows :—

If the like components of two vectors are multiplied together, the sum of the three products thus formed is equal to the scalar product of the two vectors.

When the angle between the two vectors is a right angle it is obvious that

$$A_xB_x + A_yB_y + A_zB_z = 0 \quad . \quad . \quad . \quad . \quad \textbf{(2·12)}$$

and conversely, when equation (**2·12**) holds the directions of the two vectors *must be at right angles* (if we except the trivial case where one or both of the vectors are equal to zero). If we refer the vectors **A** and **B** to new rectangular co-ordinates, in which their components are

$$A_x{}', \ A_y{}', \ A_z{}', \ \text{and} \ B_x{}', \ B_y{}', \ B_z{}',$$

the scalar product will now be

$$A_x{}'B_x{}' + A_y{}'B_y{}' + A_z{}'B_z{}'$$

and we must have

$$A_x B_x + A_y B_y + A_z B_z = A_{x}'B_{x}' + A_{y}'B_{y}' + A_{z}'B_{z}',$$

since the value of the scalar product is clearly independent of the choice of co-ordinates. We have here an example of an **invariant,** i.e. of a quantity which has the same numerical value whatever system of co-ordinates it may be referred to.

The product of the absolute values of the two vectors and the **sine** of the angle included between their directions is called their **outer** or **vector product.** In the case of the vectors **A** and **B** (Fig. 2·1) we have

$$\text{vector product} = \mathbf{AB} \sin \theta$$

We shall usually abbreviate this expression by writing it in the form   [AB].

Squaring both sides of (2·11) we have

$$\mathbf{A}^2\mathbf{B}^2 - \mathbf{A}^2\mathbf{B}^2 \sin^2 \theta = (A_x B_x + A_y B_y + A_z B_z)^2$$

or, by (2·10)

$$\mathbf{A}^2\mathbf{B}^2 \sin^2 \theta = (A_x{}^2 + A_y{}^2 + A_z{}^2)(B_x{}^2 + B_y{}^2 + B_z{}^2)$$
$$- (A_x B_x + A_y B_y + A_z B_z)^2.$$

On multiplying out, we easily recognize that this last equation is equivalent to

$$A^2 B^2 \sin^2 \theta = (A_y B_z - A_z B_y)^2 + (A_z B_x - A_x B_z)^2$$
$$+ (A_x B_y - A_y B_x)^2 \quad . \quad . \quad . \quad . \quad (2·13)$$

Obviously we may change the sign in any of the expressions $A_y B_z - A_z B_y$, etc. on the right without affecting the equation. This ambiguity is intimately associated with a corresponding feature in rectangular axes of co-ordinates, and it now becomes necessary to give a precise specification of the type of rectangular axes we propose to use. We shall do this in the following terms :

The motion of an ordinary or right-handed screw travelling along the X direction turns the Y axis towards the Z axis. In this description the letters X, Y and Z may, of course, be interchanged in a cyclic fashion. It is evident from equation (2·13) that the three quantities,

$$\sigma_x = A_y B_z - A_z B_y, \; \sigma_y = A_z B_x - A_x B_z, \; \sigma_z = A_x B_y - A_y B_x$$

can be regarded as the components of a vector the absolute value of which is $\mathbf{AB} \sin \theta$. The question arises : What is the relation between the directions of the vectors **σ, A** and **B** ? The scalar product (**σA**)

$$= \sigma_x A_x + \sigma_y A_y + \sigma_z A_z$$
$$= (A_y B_z - A_z B_y)A_x + (A_z B_x - A_x B_z)A_y + (A_x B_y - A_y B_x)A_z$$
$$= 0 \text{ identically.}$$

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It follows that the vector σ is at right-angles to $\mathbf{A}$ and by forming the scalar product $(\sigma\mathbf{B})$ we can show further that σ is also at right angles to $\mathbf{B}$. Let us turn the co-ordinate axes about the origin so that the vectors $\mathbf{A}$ and $\mathbf{B}$ lie in the XY plane in the way
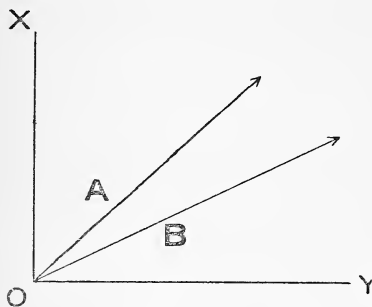
FIG. 2·11

indicated in Fig. 2·11. The components σ_x and σ_y will now be zero, and we see that σ_z is positive, since $A_xB_y - A_yB_x$ is obviously greater than zero.

This means that when the co-ordinate axes are placed in this way relatively to the vectors $\mathbf{A}$ and $\mathbf{B}$, the vector σ will be in the direction of the Z axis, and so we conclude that **the motion of an ordinary or right-handed screw travelling in the direction of σ turns the vector A towards the vector B.** We shall extend the use of the notation [**AB**] to represent the vector product completely, i.e. both in magnitude and direction. That is to say [**AB**] means the vector, the X, Y and Z components of which are respectively

$$A_yB_z - A_zB_y, \; A_zB_x - A_xB_z, \; A_xB_y - A_yB_x,$$

and [**BA**] means the vector

$$(B_yA_z - B_zA_y, \; B_zA_x - B_xA_z, \; B_xA_y - B_yA_x),$$

which has the opposite direction.

The scalar product of any vector $\mathbf{C}$ and [**AB**] is

$$(\mathbf{C}[\mathbf{AB}]) = C_x[AB]_x + C_y[AB]_y + C_z[AB]_z,$$
$$(\mathbf{C}[\mathbf{AB}]) = C_x(A_yB_z - A_zB_y) + C_y(A_zB_x - A_xB_z)$$
$$+ C_z(A_xB_y - A_yB_x),$$

or $(\mathbf{C}[\mathbf{AB}]) = \begin{vmatrix} A_x, & A_y, & A_z \\ B_x, & B_y, & B_z \\ C_x, & C_y, & C_z \end{vmatrix}$.   .   .   .   **(2·14)**

Clearly this determinant is an *invariant*, since a scalar product is an invariant. If again we imagine the axes of co-ordinates to be turned about the origin till the vectors $\mathbf{A}$ and $\mathbf{B}$ lie in the XY plane, as in Fig. (2·11), the scalar product $(\mathbf{C}[\mathbf{AB}])$ becomes $C_z[AB]_z$, since the X and Y components of [**AB**] are both zero. Therefore

$$(\mathbf{C}[\mathbf{AB}]) = \mathbf{C}\cos\varepsilon\,\mathbf{AB}\sin\theta \quad . \quad . \quad . \; \textbf{(2·15)}$$

where ε is the angle between the directions of $\mathbf{C}$ and of the Z axis. If therefore ε is less than $\dfrac{\pi}{2}$, the scalar product $(\mathbf{C}[\mathbf{AB}])$

or the determinant (2·14) is equal to the volume of the parallelopiped which is determined by the displacements A, B, C.

We may formulate this result as follows : **If the motion of an ordinary or right-handed screw travelling along the direction of C turn A towards B, then the determinant (2·14) is equal to the volume of the parallelopiped determined by the vectors A, B and C.**

Obviously we may interchange A, B and C in cyclic fashion in this theorem.

If cos ε in (2·15) is zero, the vectors C and [AB] are at right-angles to one another ; but this means that A, B *and* C *are in the same plane* and on the other hand that the determinant (2·14) is zero. In fact, if A, B and C are all different from zero, the necessary and sufficient condition that they shall be co-planar is :

$$\begin{vmatrix} A_x, A_y, A_z \\ B_x, B_y, B_z \\ C_x, C_y, C_z \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2·16)$$

§ 2·2. Co-ordinate Transformations

Let X, Y, Z and X′, Y′, Z′ be two sets of rectangular axes of co-ordinates with a common origin O ; and let P be any point, the co-ordinates of which are x, y, z and x', y', z' in the two systems respectively (Fig. 2·2). Let us further represent the cosines of the angles between X′ and X, Y, Z, by l_x, l_y and l_z respectively ; those between Y′ and X, Y, Z by m_x, m_y and m_z respectively, and so on. The problem before us is : given x, y, z, the co-ordinates of P in the system X, Y, Z,

Fig. 2·2

to find x', y', z', its co-ordinates in the other system X′, Y′, Z′, and vice versa. Drop a perpendicular Pm on OX′, so that Om is equal to x', the X′ co-ordinate of P in the system X′, Y′, Z′. We may regard both OP and Om as vectors and we have clearly

$$\begin{array}{l} \text{OP} \equiv (x,\ y,\ z) \\ \text{O}m \equiv (x'l_x,\ x'l_y,\ x'l_z) \end{array} \Big\} \text{System X, Y, Z.}$$

$$\begin{array}{l} \text{OP} \equiv (x',\ y',\ z') \\ \text{O}m \equiv (x',\ o,\ o) \end{array} \Big\} \text{System X′, Y′, Z′.}$$

The rule (2·11) gives us for their scalar product the alternative expressions,
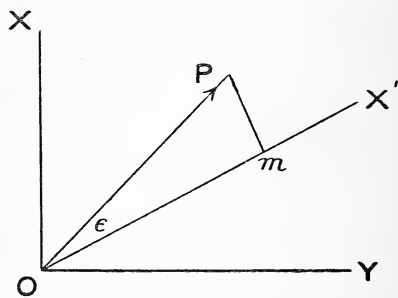
$$x'l_x x + x'l_y y + x'l_z z \text{ (System X, Y, Z)}$$

and
$$x'^2 \text{ (System X', Y', Z').}$$

On equating these two expressions, and dividing by the common factor x', we finally obtain

$$x' = l_x x + l_y y + l_z z.$$

In a similar way we may show that

$$\begin{aligned} y' &= m_x x + m_y y + m_z z, \\ z' &= n_x x + n_y y + n z_z. \end{aligned} \quad \cdots \quad \textbf{(2·2)}$$

The equations of the inverse transformation are easily found to be

$$\begin{aligned} x &= l_x x' + m_x y' + n_x z' \\ y &= l_y x' + m_y y' + n_y z' \\ z &= l_z x' + m_z y' + n_z z'. \end{aligned} \quad \cdots \quad \textbf{(2·21)}$$

We may, evidently, regard l_x, l_y and l_z as the components of a unit vector (i.e. a vector the absolute value of which is unity) in the system X, Y, Z. A similar remark applies to (m_x, m_y, m_z) and (n_x, n_y, n_z). And in the system X', Y', Z' we may regard (l_x, m_x, n_x), (l_y, m_y, n_y) and (l_z, m_z, n_z) as unit vectors.

For many purposes it is convenient to represent these direction cosines by a single letter, distinguishing one from another by numerical subscripts, thus :

$$\begin{aligned} (l_x, \ l_y, \ l_z) &\equiv (a_{11}, \ a_{12}, \ a_{13}), \\ (m_x, \ m_y, \ m_z) &\equiv (a_{21}, \ a_{22}, \ a_{23}), \\ (n_x, \ n_y, \ n_z) &\equiv (a_{31}, \ a_{32}, \ a_{33}). \end{aligned}$$

All six equations of transformation given above are conveniently represented in the following schematic form :—

	x	y	z
x'	a_{11}	a_{12}	a_{13}
y'	a_{21}	a_{22}	a_{23}
z'	a_{31}	a_{32}	a_{33}

$$\cdots \quad \textbf{(2·22)}$$

Mathematically a vector may be defined as a **set of three quantities which transform according to the rules embodied in (2·22).**

There are certain important and interesting relations between the direction cosines a. For example, the sum of the squares of the a's in any horizontal row, or in any vertical column of **(2·22)** is equal to unity :

$$a_{11}{}^2 + a_{12}{}^2 + a_{13}{}^2 = 1,$$

etc.
$$a_{12}{}^2 + a_{22}{}^2 + a_{32}{}^2 = 1 \quad \cdots \quad \textbf{(2·23)}$$

The correctness of these equations is obvious, since in each case the left-hand member can be regarded as the sum of the squares of the components of a unit vector. Further, the sum of the products of corresponding a's in any two horizontal rows, or in any two vertical columns is zero, e.g.,

$$a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0,$$
$$a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0 \quad . \quad . \quad . \text{ (2·24)}$$

and so on. These equations follow since the left-hand member in each case can be regarded as the scalar product of two unit vectors which are at right angles to one another.

Finally we have the relation

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} = 1 \quad . \quad . \quad . \quad . \quad . \text{ (2·25)}$$

since by **(2·14)** this determinant represents the volume of the parallelopiped bounded by the three mutually perpendicular unit vectors (l_x, l_y, l_z), (m_x, m_y, m_z) and (n_x, n_y, n_z).

§ 2·3. Tensors of Higher Rank

We are now able to define more precisely a tensor of higher rank than a vector. Take, for example, a tensor of the second rank, such as that which expresses the state of stress in an elastic solid. It is a set of 3^2 quantities, called its components,

$$p_{xx}, \quad p_{xy}, \quad p_{xz},$$
$$p_{yx}, \quad p_{yy}, \quad p_{yz},$$
$$p_{zx}, \quad p_{zy}, \quad p_{zz},$$

having the property that the values of the components p_{xx}', p_{xy}', etc., in the system X'Y'Z' are calculated from p_{xx}, p_{xy}, etc., the components in the system X, Y, Z, by precisely the same rules as those for calculating $A_x'B_x'$, $A_x'B_y'$, etc., from the products A_xB_x, A_xB_y, etc., where A_x, A_y, B_y, etc., are the components of two vectors. A tensor of the second rank is said to be symmetrical when the subscripts of a component may be interchanged, e.g., when

$$p_{xy} = p_{yx}$$

The system of stresses in an elastic solid in equilibrium constitutes such a symmetrical tensor. If, on the other hand,

$$p_{xy} = -p_{yx},$$

the tensor is said to be anti-symmetrical. Since in this case,

$$p_{xx} = -p_{xx},$$

the components p_{xx}, p_{yy}, etc., with two like subscripts will all three be zero. As an example of an anti-symmetrical tensor we

may instance that formed from two vectors A and B in the following way :—

$$A_xB_x - A_xB_x, \quad A_xB_y - A_yB_x, \quad A_xB_z - A_zB_x,$$
$$A_yB_x - A_xB_y, \quad A_yB_y - A_yB_y, \quad A_yB_z - A_zB_y,$$
$$A_zB_x - A_xB_z, \quad A_zB_y - A_yB_z, \quad A_zB_z - A_zB_z.$$

Its XX, YY and ZZ components are zero and the remaining six are the components of the vectors [**AB**] and [**BA**]. In fact we may dispense with these vector products by employing this tensor.

More generally, if we have n vectors and select one component of each and multiply them together, the 3^n products obtained from all the possible selections constitute a typical tensor of the nth rank, and any set of 3^n quantities will constitute a tensor of rank n if they obey the same laws of transformation as the 3^n components of the typical tensor.

§ 2·4. Vector and Tensor Fields

We shall often be concerned with regions in which electric, magnetic or gravitational forces manifest themselves. We call such regions **fields of force.** They are characterized in each of these examples by a vector which varies continuously from point to point in the region and which may be termed the **intensity** of the field. In hydrodynamics we are concerned with regions filled with a fluid, the motion of which can be described by giving its velocity at every point in the region. In all these examples we may use the general term **vector field** for the region in question. Or we may be concerned (e.g. when we are studying the state of stress in an elastic solid, or the Maxwell stresses in an electrostatic field) with the components of a tensor of higher rank than a vector and with the way in which they vary from one point in the field to another. In such a case we may call the region a **tensor field.**

The description and investigation of vector or tensor fields involves the use of partial differential equations, and we shall therefore study some of the features of the operations $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, where the *round d*'s are the conventional symbols for partial differentiation, i.e. $\frac{\partial}{\partial x}$ means a differentiation in which the other independent variables y, z and the time are kept constant. In the first place we may show that $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ have the same transformation properties as the X, Y, Z components of a vector.

Let ϕ be any quantity which varies continuously from point to point. Then by a well-known theorem of the differential calculus,

$$\frac{\partial \phi}{\partial x'} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial x'} \qquad . \quad . \quad (2·4)$$

But by (2·22)

$$x = a_{11}x' + a_{21}y' + a_{31}z',$$
$$y = a_{12}x' + a_{22}y' + a_{32}z',$$
$$z = a_{13}x' + a_{23}y' + a_{33}z',$$

therefore

$$\frac{\partial x}{\partial x'} = a_{11}, \quad \frac{\partial y}{\partial x'} = a_{12}, \quad \frac{\partial z}{\partial x'} = a_{13},$$

and on substituting in (2·4) we have

$$\frac{\partial \phi}{\partial x'} = a_{11}\frac{\partial \phi}{\partial x} + a_{12}\frac{\partial \phi}{\partial y} + a_{13}\frac{\partial \phi}{\partial z},$$

or, dropping ϕ, we have the equivalence,

$$\frac{\partial}{\partial x'} \equiv a_{11}\frac{\partial}{\partial x} + a_{12}\frac{\partial}{\partial y} + a_{13}\frac{\partial}{\partial z}. \quad . \quad . \quad . \quad (2·41)$$

This is sufficient to establish the vectorial character of these operations.

If $\mathbf{A} \equiv (A_x, A_y, A_z)$ is a field vector, the quantity

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

will be an invariant since it has the same transformation properties as a scalar product. It is called the **divergence** of the vector $\mathbf{A}$ and is written

<p style="text-align:center">div A.</p>

Furthermore the three quantities

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \quad \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \quad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y},$$

must be the X, Y and Z components of a vector, since they have the same transformation properties as the components of a vector product. This vector is called the **curl** of A or the **rotation** of A and is written

<p style="text-align:center">curl A or rot A.</p>

It is easy to show that

$$\text{div curl } \mathbf{A} = 0, \quad . \quad . \quad . \quad . \quad (2·42)$$

where $\mathbf{A}$ is any field vector. We have in fact,

$$\mathbf{div\ curl\ A} = \frac{\partial}{\partial x}\{\text{curl A}\}_x + \frac{\partial}{\partial y}\{\text{curl A}\}_y + \frac{\partial}{\partial z}\{\text{curl A}\}_z$$

or   **div curl A** $= \dfrac{\partial}{\partial x}\left(\dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}\right)$

$$+ \dfrac{\partial}{\partial z}\left(\dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}\right).$$

It will be seen that this is identically zero.

If $\mathbf{A} \equiv (A_x,\ A_y,\ A_z)$ is a vector and ϕ any *scalar* quantity, it is obvious that

$$(A_x\phi,\ A_y\phi,\ A_z\phi)$$

is a vector.   Similarly

$$\left(\dfrac{\partial\phi}{\partial x},\ \dfrac{\partial\phi}{\partial y},\ \dfrac{\partial\phi}{\partial z}\right)$$

is a vector.   Such a vector is called the **gradient** of the scalar quantity and is written **grad ϕ**.   We have therefore

$$\mathbf{grad}\ \boldsymbol{\phi} \equiv \left(\dfrac{\partial\phi}{\partial x},\ \dfrac{\partial\phi}{\partial y},\ \dfrac{\partial\phi}{\partial z}\right) \quad . \quad . \quad . \quad . \quad \textbf{(2·43)}$$

The components of the vector **curl grad** ϕ are all identically zero.   Take the X component for example :

$$\{\text{curl grad } \phi\}_x = \dfrac{\partial}{\partial y}\{\text{grad } \phi\}_z - \dfrac{\partial}{\partial z}\{\text{grad } \phi\}_y$$

or $\{\text{curl grad } \phi\}_x = \dfrac{\partial}{\partial y}\left\{\dfrac{\partial\phi}{\partial z}\right\} - \dfrac{\partial}{\partial z}\left\{\dfrac{\partial\phi}{\partial y}\right\}$

$$= 0 \text{ identically.}$$

We may write this result in the form

$$\mathbf{curl\ grad}\ \phi = 0 \ . \quad . \quad . \quad . \quad . \quad \textbf{(2·431)}$$

The quasi-vectorial character of the operations $\dfrac{\partial}{\partial x},\ \dfrac{\partial}{\partial y},\ \dfrac{\partial}{\partial z}$ makes it often a convenience to represent them by the symbolism used for vectors.   We shall frequently denote them by the symbol $\boldsymbol{\nabla}$ (pronounced *nabla*), thus

$$\boldsymbol{\nabla} \equiv (\nabla_x,\ \nabla_y,\ \nabla_z) \equiv \left(\dfrac{\partial}{\partial x},\ \dfrac{\partial}{\partial y},\ \dfrac{\partial}{\partial z}\right)$$

and therefore

$$\mathbf{grad}\ \phi \equiv (\nabla_x\phi,\ \nabla_y\phi,\ \nabla_z\phi,)$$

or $\mathbf{grad}\ \phi \equiv \boldsymbol{\nabla}\phi,$

and   **div A** $\equiv \nabla_x A_x + \nabla_y A_y + \nabla_z A_z \equiv (\boldsymbol{\nabla}\mathbf{A}).$

The quantity

$$\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$$

is called the **laplacian** of the scalar ϕ in honour of the great French mathematician Laplace.   Our notation enables us to represent it by $\boldsymbol{\nabla}^2\phi.$

A useful formula, frequently used in electromagnetic theory, is the following :—

$$\text{div } [AB] = (B \text{ curl } A) - (A \text{ curl } B) \quad . \quad (2\cdot44)$$

This can be proved by writing out **div [AB]** in full.

$$\text{div } [AB] = \frac{\partial}{\partial x}[AB]_x + \frac{\partial}{\partial y}[AB]_y + \frac{\partial}{\partial z}[AB]_z.$$

The first of the three terms on the right expands to

$$A_y\frac{\partial B_z}{\partial x} + B_z\frac{\partial A_y}{\partial x} - A_z\frac{\partial B_y}{\partial x} - B_y\frac{\partial A_z}{\partial x},$$

and the remaining two terms on the right give

$$A_z^{\mathbf{!}}\frac{\partial B_x}{\partial y} + \overset{o}{B}_x\frac{\partial A_z}{\partial y} - A_x\frac{\partial B_z}{\partial y} - B_z\frac{\partial A_x}{\partial y}$$

$$+ A_x\frac{\partial B_y}{\partial z} + B_y\frac{\partial A_x}{\partial z} - A_y\frac{\partial B_x}{\partial z} - \overset{o}{B}_x\frac{\partial A_y}{\partial z}.$$

The pair of terms, marked o, taken together make

$$B_x\{\text{curl } A\}_x.$$

Similarly, we find a pair of terms equivalent to

$$B_y\{\text{curl } A\}_y,$$

and another equivalent to

$$B_z\{\text{curl } A\}_z,$$

so that six of the terms make up

$$(B \text{ curl } A).$$

In the same way the remaining six are seen to make up

$$- (A \text{ curl } B)$$

and thus the formula is established.

An equally important formula is :

$$\text{curl curl } A = \text{grad div } A - \nabla^2 A \quad . \quad . \quad (2\cdot45)$$

which we can likewise establish by writing out the left-hand side, or the X component of the left-hand side, in full.

$$\{\text{curl curl } A\}_x = \frac{\partial}{\partial y}\left\{\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right\} - \frac{\partial}{\partial z}\left\{\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right\},$$

or $\quad \{\text{curl curl } A\}_x = \frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \left(\frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right),$

and if we add and subtract $\frac{\partial^2 A_x}{\partial x^2}$ on the right-hand side we get

$$\{\text{curl curl } A\}_x = \frac{\partial}{\partial x} \text{ div } A - \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right),$$

a result which may be expressed in the form (**2·45**).

CHAPTER II

THE THEOREMS OF GAUSS, GREEN AND STOKES. FOURIER'S EXPANSION

§ 3. THEOREM OF GAUSS

IMAGINE a closed surface, *abc*, Fig. 3, and a field vector (A_x, A_y, A_z) which varies continuously throughout the volume enclosed by it. We shall investigate the integral

$$\iiint \mathrm{div}\ \mathbf{A}\ dxdydz \quad . \quad . \quad . \quad (3)$$

It is important to grasp the precise meaning of this integral.

X

$(dS)_2$ ϕ_2

b

a

ϕ_1

c

$(dS)_1$

O — Y
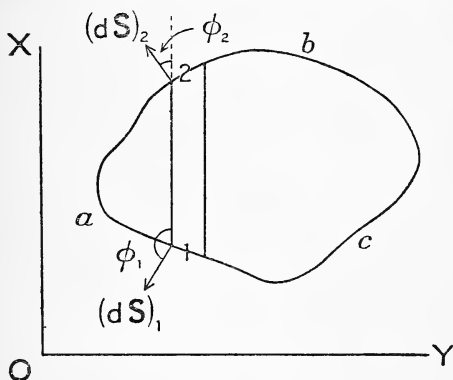
Fig. 3

We suppose the whole volume *abc* divided into small elements and each element of volume multiplied by the value of **div A** at some point within it. The integral (3) is the limit to which the sum of all the products so formed approximates as the elements of volume become indefinitely small. It is not essential that the elements of volume should be rectangular, or that the sum should be expressed by the use of the *triple* symbol of integration. We may write (3) in the form

$$\int \mathrm{div}\ A\ dv \quad . \quad . \quad . \quad . \quad (3\cdot001)$$

where *dv* represents an element of volume of any shape. It is convenient, however, to use the triple symbol when we wish to draw attention to the 3-dimensional character of the region over which the integration extends.

From the definition of **div A** (§ 2·4) we have for (3)

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$$\iiint \left\{ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right\} dx\, dy\, dz \quad . \quad . \text{ (3·002)}$$

so that it may be treated as the sum of three integrals. We shall begin with

$$\iiint \frac{\partial A_x}{\partial x} dx\, dy\, dz,$$

and carry out the integration or summation over all the elements of volume from 1 to 2 (Fig. 3) in a single narrow vertical column with the uniform horizontal cross-section $dy\, dz$. For this restricted volume we have

$$\iiint \frac{\partial A_x}{\partial x} dx\, dy\, dz = dy\, dz \int_1^2 \frac{\partial A_x}{\partial x} dx$$

$$= dy\, dz \{ (A_x)_2 - (A_x)_1 \} \quad . \quad . \quad . \text{ (3·003)}$$

where $(A_x)_1$ and $(A_x)_2$ are the values of A_x at the terminal points 1 and 2 respectively, where the vertical column cuts the surface *abc*. Let the elements of area at the two ends of the vertical column be $(\mathbf{dS})_1$ and $(\mathbf{dS})_2$. It is helpful to imagine short perpendiculars erected on the surface at the points 1 and 2 and directed outwards, each perpendicular having a length equal to the area $\mathbf{dS}$ of the corresponding element ; (see $(\mathbf{dS})_1$ and $(\mathbf{dS})_2$ in Fig. 3). These perpendiculars may be regarded as vectors with the absolute values $(\mathbf{dS})_1$ and $(\mathbf{dS})_2$. Let ϕ_1 and ϕ_2 be the angles between the directions of the vectors $(\mathbf{dS})_1$ and $(\mathbf{dS})_2$ respectively and the X axis. We have then

$$(\mathbf{dS})_2 \cos \phi_2 = dy\, dz,$$
$$- (\mathbf{dS})_1 \cos \phi_1 = dy\, dz,$$

or

$$(dS_x)_2 = dy\, dz,$$
$$- (dS_x)_1 = dy\, dz.$$

Substituting in (3·003) we get

$$dy\, dz \int_1^2 \frac{\partial A_x}{\partial x} dx = (A_x dS_x)_2 + (A_x dS_x)_1$$

or, otherwise expressed, the integral

$$\iiint \frac{\partial A_x}{\partial x} dx\, dy\, dz,$$

when extended over such a vertical column, is equal to the sum of the products $A_x dS_x$, where the surface *abc* is cut by the column. When the integral is extended over the *whole volume* *abc*, i.e. over all the vertical columns in it, we get the sum

of the products $A_x dS_x$ for all the elements of area making up the surface.

Therefore

$$\iiint \frac{\partial A_x}{\partial x}\, dx\, dy\, dz = \iint A_x dS_x,$$

where the summation on the left extends over the whole volume *abc*, and that on the right over the whole surface *abc*.   Similarly we have,

$$\iiint \frac{\partial A_y}{\partial y}\, dx\, dy\, dz = \iint A_y dS_y,$$

and

$$\iiint \frac{\partial A_z}{\partial z}\, dx\, dy\, dz = \iint A_z dS_z.$$

Adding these three equations, we get

$$\iiint \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dx\, dy\, dz$$

$$= \iint \{A_x dS_x + A_y dS_y + A_z dS_z\}$$

or

$$\boxed{\iiint \mathbf{div\ A}\ dx\, dy\, dz = \iint (\mathbf{A\ dS})}$$.   .   (**3·01**)

where (**A dS**) on the right hand is the scalar product of the vectors $\mathbf{A} \equiv (A_x, A_y, A_z)$ and $\mathbf{dS} \equiv (dS_x,\ dS_y,\ dS_z)$.   Equation (**3·01**) expresses the **theorem of Gauss.**

§ 3·1.   Green's Theorem

Let the vector **A** in (**3·01**) have the form

$$U \ \dot{\mathrm{g}}\mathrm{rad}\ V,$$

where U and V are scalars, which, with their first and second differential quotients, are continuous functions of x, y and z in the volume *abc*.   We thus have

$$\iiint \mathbf{div}\ \{U \ \dot{\mathrm{g}}\mathrm{rad}\ V\}\, dx\, dy\, dz = \iint (U \ \dot{\mathrm{g}}\mathrm{rad}\ V,\ \mathbf{dS}),$$

or $\iiint U \nabla^2 V\, dx\, dy\, dz + \iiint (\dot{\mathrm{g}}\mathrm{rad}\ U, \dot{\mathrm{g}}\mathrm{rad}\ V)\, dx\, dy\, dz$

$$= \iint (U \ \dot{\mathrm{g}}\mathrm{rad}\ V,\ \mathbf{dS})$$.   (**3·1**)

Interchanging U and V in (**3·1**), we get

$$\iiint V \nabla^2 U\, dx\, dy\, dz + \iiint (\dot{\mathrm{g}}\mathrm{rad}\ U, \dot{\mathrm{g}}\mathrm{rad}\ V)\, dx\, dy\, dz$$

$$= \iint (V \ \dot{\mathrm{g}}\mathrm{rad}\ U,\ \mathbf{dS})$$.   .   (**3·11**)

and on subtracting $(3\cdot11)$ from $(3\cdot1)$ we obtain

$$\iiint \{U\nabla^2 V - V\nabla^2 U\}\, dx\, dy\, dz$$
$$= \iint (U\ \mathbf{grad}\ V,\ \mathbf{dS}) - \iint (V\ \mathbf{grad}\ U,\ \mathbf{dS})\quad (3\cdot12)$$

This result, known as **Green's theorem,** was published in 1828 by George Green in an epoch-making work entitled *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.* If we represent distances measured in the direction of an outward normal to the surface *abc* by the letter *n*, the normal component of $\mathbf{grad}\ V$, i.e. the product of $\mathbf{grad}\ V$ and the cosine of the angle between its direction and that of $\mathbf{dS}$ or of the normal, is

$$\frac{\partial V}{\partial n},$$

so that $(3\cdot12)$ may be written in the form

$$\iiint \{U\nabla^2 V - V\nabla^2 U\}\, dx\, dy\, dz$$
$$= \iint \left\{ U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n} \right\} \mathbf{dS} \qquad . \quad . \quad . \quad (3\cdot13)$$

In this equation let the value of U at any point be equal to $\frac{1}{r}$ where r is the distance of the point from the origin. If then the origin is *outside* the volume *abc* over which the triple integral is extended, we have from $(3\cdot13)$

$$\iiint \frac{1}{r}\nabla^2 V\, dx\, dy\, dz = \iint \left\{ \frac{1}{r}\frac{\partial V}{\partial n} - V\frac{\partial \left(\frac{1}{r}\right)}{\partial n} \right\} \mathbf{dS}\ . \quad (3\cdot14)$$

Since $\nabla^2\frac{1}{r}$ can be shown, as follows, to be zero. We have namely

$$\frac{\partial \left(\frac{1}{r}\right)}{\partial x} = -\frac{1}{r^2}\cdot\frac{\partial r}{\partial x},$$

therefore
$$\frac{\partial^2 \left(\frac{1}{r}\right)}{\partial x^2} = \frac{2}{r^3}\left(\frac{\partial r}{\partial x}\right)^2 - \frac{1}{r^2}\frac{\partial^2 r}{\partial x^2}. \quad . \quad . \quad . \quad (3\cdot141)$$

Now
$$r^2 = x^2 + y^2 + z^2,$$

therefore
$$2r\frac{\partial r}{\partial x} = 2x$$

or
$$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

Further
$$\frac{\partial^2 r}{\partial x^2} = = \frac{x^2}{r^3} + \frac{1}{r}.$$

Substituting these expressions for $\frac{\partial r}{\partial x}$ and $\frac{\partial^2 r}{\partial x^2}$ in (3·141) we get

$$\frac{\partial^2\left(\frac{1}{r}\right)}{\partial x^2} = \frac{2x^2}{r^5} - \frac{1}{r^2}\left(\frac{1}{r} - \frac{x^2}{r^3}\right),$$

or

$$\frac{\partial^2\left(\frac{1}{r}\right)}{\partial x^2} = \frac{3x^2}{r^5} - \frac{1}{r^3}.$$

Similarly

$$\frac{\partial^2\left(\frac{1}{r}\right)}{\partial y^2} = \frac{3y^2}{r^5} - \frac{1}{r^3},$$

and

$$\frac{\partial^2\left(\frac{1}{r}\right)}{\partial z^2} = \frac{3z^2}{r^5} - \frac{1}{r^3}.$$

On adding the last three equations, we find

$$\nabla^2\left(\frac{1}{r}\right) = \frac{\partial^2\left(\frac{1}{r}\right)}{\partial x^2} + \frac{\partial^2\left(\frac{1}{r}\right)}{\partial y^2} + \frac{\partial^2\left(\frac{1}{r}\right)}{\partial z^2}$$

$$= \frac{3(x^2 + y^2 + z^2)}{r^5} - \frac{3}{r^3} = 0.$$

Let us apply (3·14) to the case where the volume integration
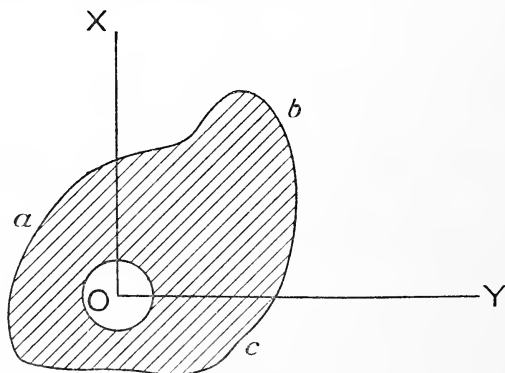
FIG. 3·1

extends over a region like that indicated by the shaded part of the diagram in Fig. 3·1. This region is enclosed between the

surface abc and the surface of a sphere of small radius, R, having the origin for its centre.   The surface integration is now extended over the surface abc and over the surface of the small sphere as well.

At points on this latter surface

$$\frac{\partial V}{\partial n} = -\frac{\partial V}{\partial r},$$

since the direction of the outward normal is exactly opposite to that of r.   Similarly

$$-\frac{\partial \left(\frac{1}{r}\right)}{\partial n} = -\frac{\partial \frac{1}{r}}{\partial r} = \frac{1}{R^2}$$

Therefore the part of the surface integral of (3·14) extended over the small sphere may be expressed as

$$-\frac{\overline{\partial V}}{\partial r}\iint \frac{dS}{R} - \overline{V}\iint \frac{dS}{R^2},$$

where $\dfrac{\overline{\partial V}}{\partial r}$ and $\overline{V}$ are average values of $\dfrac{\partial V}{\partial r}$ and V respectively over the surface of the sphere.   This part of the surface integral is therefore equal to

$$-4\pi R\frac{\overline{\partial V}}{\partial r} - 4\pi\overline{V},$$

and since $\dfrac{\overline{\partial V}}{\partial r}$ and $\overline{V}$ are continuous it will approach the limit

$$-4\pi V_0$$

as R approaches zero, if V_0 is the value of V at the origin. We have therefore

$$\iiint \frac{1}{r}\nabla^2 V \, dx \, dy \, dz = \iint \left\{\frac{1}{r}\frac{\partial V}{\partial n} - V\frac{\partial \left(\frac{1}{r}\right)}{\partial n}\right\}dS - 4\pi V_0,$$

$$\text{or } V_0 = -\frac{1}{4\pi}\iiint \frac{1}{r}\cdot\nabla^2 V \, dx \, dy \, dz + \frac{1}{4\pi}\iint \left\{\frac{1}{r}\frac{\partial V}{\partial n}\right.$$

$$\left. - V\frac{\partial \left(\frac{1}{r}\right)}{\partial n}\right\}dS \quad . \quad . \quad . \quad (3·15)$$

In this formula the surface integral is extended over the outer surface abc of Fig. 3·1, and it is understood that the volume integral now means not merely the result of integrating over the

shaded volume, but the limit approached by this integral when R approaches the limit zero.

Imagine the surface abc to be enlarged, so that the distance r of any point on it from the origin approaches infinity, or so that $\frac{1}{r}$ approaches zero; then it may happen that the surface integral also approaches zero in the limit. It is easily seen that this must happen when V diminishes in the same way as $\frac{1}{r}$ at great distances from the origin, that is to say, when the product rV never exceeds some finite number, however great r may be. For

$$\frac{1}{r}\frac{\partial V}{\partial n} \quad \text{and} \quad V\frac{\partial\left(\frac{1}{r}\right)}{\partial n}$$

are both of the order of magnitude of $\frac{1}{r^3}$, whereas the area of the surface is of the order of r^2. In such a case (**3·15**) becomes

$$V_0 = -\frac{1}{4\pi}\iiint \frac{\nabla^2 V}{r}\,dx\,dy\,dz \qquad . \qquad . \qquad . \ (\mathbf{3 \cdot 16})$$

where the integration is extended over all space.

§ 3·2. Extensions of the Theorems of Gauss and Green

If **A, B** and **C** are three vectors, the quantity

$$C_x A_x B_x + C_y A_x B_y + C_z A_x B_z$$

is the X component of a vector, since

$$C_x B_x + C_y B_y + C_z B_z = (\mathbf{CB})$$

is a scalar or invariant quantity. Now $A_x B_x$, $A_x B_y$ and $A_x B_z$ are the XX, XY and XZ components of a tensor of the second rank and it follows that

$$C_x T_{xx} + C_y T_{xy} + C_z T_{xz}$$

is also the X component of a vector, if **C** is any vector and T_{xx} etc. any tensor of the second rank. This will be understood when it is remembered that the components of tensors are defined mathematically by their transformation properties (see §§ 2·2, 2·3). We may similarly infer that

$$\nabla_x T_{xx} + \nabla_y T_{xy} + \nabla_z T_{xz},$$

or

$$\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z},$$

is the X component of a vector and

$$\frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z}$$

and
$$\frac{\partial T_{xx}}{\partial x} + \frac{\delta T_{xy}}{\partial y} + \frac{\delta T_{xz}}{\partial z}$$

are respectively the Y and Z components of the same vector. It is usual to extend the scope of the term *divergence* to include this vector. Therefore

$$\mathbf{div\ T} \equiv \left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z}, \frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z}, \right.$$
$$\left. \frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{zy}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \right) \quad . \quad . \quad . \quad . \quad \textbf{(3·2)}$$

The method employed to deduce the theorem of Gauss can be applied to prove the statement :—

$$\iiint \left\{ \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \right\} dx\, dy\, dz$$
$$= \iint \{ T_{xx}dS_x + T_{xy}dS_y + T_{xz}dS_z \} \quad . \quad . \quad \textbf{(3·21)}$$

Green's theorem, and the formulæ deduced from it, naturally admit of a similar extension. We can, for example, deduce the equation

$$F_{x0} = - \frac{1}{4\pi} \iiint \frac{\nabla^2 F_x}{r} dx\, dy\, dz \quad . \quad . \quad . \quad \textbf{(3·22)}$$

which corresponds to **(3·16)** and in which F_{x0} means the value of F_x at the point $r = 0$. The validity of this formula is subject of course to conditions strictly analogous to those which apply in the case of **(3·16)**.

§ 3·3. Theorem of Stokes

It has been shown already **(2·42)** that
$$\mathbf{div\ curl\ A}$$
is identically zero and therefore
$$\iiint \mathbf{div\ curl\ A}\ dx\, dy\, dz = 0,$$

the integration being extended over any volume within which the vector $\mathbf{A}$ and its first derivatives are continuous. Now applying the theorem of Gauss we get

$$\iint (\mathbf{curl\ A, dS}) = 0 \quad . \quad . \quad . \quad . \quad \textbf{(3·3)}$$

the integration being now extended over the bounding surface (*abcd*, Fig. 3·3). Imagine the surface *abcd* to be divided into

3

two parts by the closed loop $\alpha\beta\gamma\delta$. Equation (3·3) may be written

$$\iint\limits_{a\,b} (\text{curl } \mathbf{A},\ d\mathbf{S}) + \iint\limits_{c\,d} (\text{curl } \mathbf{A},\ d\mathbf{S}) = 0 \quad . \ (3\text{·}301)$$

Where ab indicates the part of the integral over the portion ab of the surface to the left of the loop $\alpha\beta\gamma\delta$ and cd the part extended over the portion to the right of the loop. Now suppose
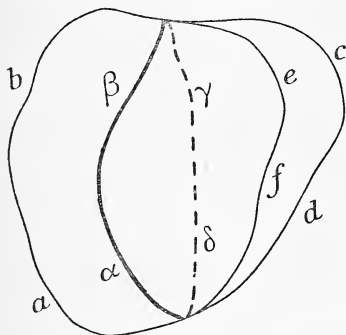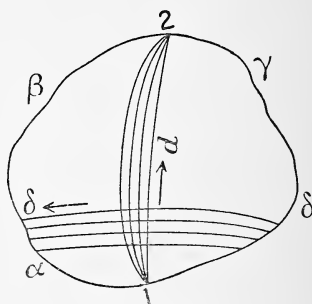
FIG. 3·3   FIG. 3·31

the surface cd to be replaced by another surface ef with the same boundary line $\alpha\beta\gamma\delta$. We shall have

$$\iint\limits_{a\,b} (\text{curl } \mathbf{A},\ d\mathbf{S}) + \iint\limits_{e\,f} (\text{curl } \mathbf{A},\ d\mathbf{S}) = 0 \qquad (3\text{·}302)$$

From (3·301) and (3·302) we have

$$\iint\limits_{c\,d} (\text{curl } \mathbf{A}, d\mathbf{S}) = \iint\limits_{e\,f} (\text{curl } \mathbf{A}, d\mathbf{S}),$$

and therefore the value of the integral can only depend on the values of the vector $\mathbf{A}$ along the curve $\alpha\beta\gamma\delta$ which forms the boundary of the surface cd or ef. This suggests the problem of expressing $\iint (\text{curl } \mathbf{A}, d\mathbf{S})$ in terms of what is given for points on the boundary $\alpha\beta\gamma\delta$ (Fig. 3·31) of the surface. Let us construct two sets of lines on the surface, each set containing an infinite number of lines. The first set, which we shall call the **d** lines, all begin at a common point (1 in Fig. 3·31) and all end at a common point (2 in Fig. 3·31). We shall suppose them to be sensibly parallel to one another in any small neighbourhood. The element of area between any two adjacent **d** lines may be called a **d** area. The second set of lines, which may be termed **δ** lines, are so drawn as to divide the **d** areas

into infinitesimal parallelograms, the area of any one of which may be symbolized by **dS**. The increment of any quantity ϕ, as we travel along a **d** line in the direction 1 to 2 (shown in Fig. 3·31 by an arrow), from one **δ** line to the next, will be represented by $d\phi$. In the same way $\delta\phi$ will represent the increment of ϕ which occurs in travelling along a **δ** line, in the direction indicated by the arrow, from one **d** line to the next. If the letter l be used for distances measured along any of these lines, an element of area **dS** will be equal to $\delta l . dl . \sin\theta$ (see Fig. 3·32). As usual we shall regard **dS** as a vector and write

FIG. 3·32

$$\mathbf{dS} = [\boldsymbol{\delta l}, \mathbf{dl}]$$
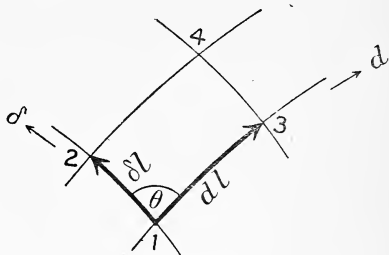
We can visualize **dS** as a short displacement perpendicular to the surface of the element and directed away from the reader. We evidently have

$$\begin{aligned} dS_x &= \delta y\, dz - \delta z\, dy, \\ dS_y &= \delta z\, dx - \delta x\, dz, \\ dS_z &= \delta x\, dy - \delta y\, dx. \end{aligned} \quad\quad \text{(3·31)}$$

In these equations

$$\begin{aligned} \boldsymbol{\delta l} &\equiv (\delta x,\ \delta y,\ \delta z) \\ \mathbf{dl} &\equiv (dx,\ dy,\ dz) \end{aligned} \quad\quad \text{(3·311)}$$

Let the X co-ordinate of the point 1 (Fig. 3·32) be x; the X co-ordinate of the point 3 will be $x + dx$, and as we pass from 3 to 4 we realize that the X co-ordinate of the point 4 must be

$$x + dx + \delta(x + dx),$$

or $\qquad\qquad x + dx + \delta x + \delta dx \quad . \quad . \quad . \quad \text{(3·312)}$

If we travel from the point 1 to the point 4 by way of the point 2, we find the X co-ordinate of 2 to be $x + \delta x$ and that of 4 to be

$$x + \delta x + d(x + \delta x),$$

or $\qquad\qquad x + \delta x + dx + d\delta x \quad . \quad . \quad . \quad \text{(3·313)}$

Both of the expressions (3·312) and (3·313) represent the X co-ordinate of the same point, and it follows that

$$\delta dx = d\delta x \quad . \quad . \quad . \quad . \quad \text{(3·314)}$$

This means that the operations δ and d are interchangeable, at any rate when applied to the co-ordinates.

The integral $\iint (\mathbf{curl\ A, dS})$ over a surface bounded by

the line $\alpha\beta\gamma\delta$ may be treated as a sum of integrals, each extended over a **d** area. A typical **d** area is shown in Fig. 3·33. The surface integral over it may be written
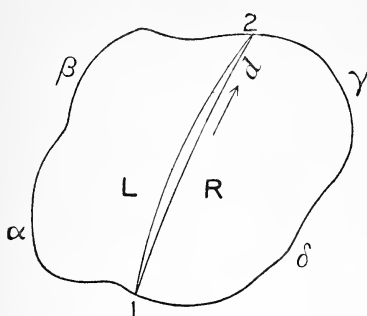
FIG. 3·33

$$\int \{[\operatorname{curl} A]_x dS_x + [\operatorname{curl} A]_y dS_y + [\operatorname{curl} A]_z dS_z\},$$

in which the *double* symbol of integration has been dropped, since we are now dealing with the sum of a singly infinite set of elements extending over the **d** area from 1 to 2. Suppose the integrand to be written out in full, using the definition of **curl** in § 2·4 and equations (3·31). The part of the integral involving A_x only is

$$\int_1^2 \left[\frac{\partial A_x}{\partial z}(\delta z\, dx - \delta x\, dz) - \frac{\partial A_x}{\partial y}(\delta x\, dy - \delta y\, dx) \right],$$

$$= \int_1^2 \left[\left\{\frac{\partial A_x}{\partial y}\delta y + \frac{\partial A_x}{\partial z}\delta z\right\} dx - \left\{\frac{\partial A_x}{\partial y}dy + \frac{\partial A_x}{\partial z}dz\right\} \delta x \right].$$

Adding and subtracting

$$\int_1^2 \frac{\partial A_x}{\partial x}\delta x\, dx,$$

we get for this part of the integral

$$\int_1^2 \left[\left\{\frac{\partial A_x}{\partial x}\delta x + \frac{\partial A_x}{\partial y}\delta y + \frac{\partial A_x}{\partial z}\delta z\right\} dx - \left\{\frac{\partial A_x}{\partial x}dx + \frac{\partial A_x}{\partial y}dy + \frac{\partial A_x}{\partial z}dz\right\} \delta x \right]$$

$$= \int_1^2 [\delta A_x dx - dA_x \delta x].$$

Now add to this result the integral

$$\int_1^2 d(A_x \delta x)$$

which is equal to zero, since δx vanishes at the points 1 and 2.

We have therefore for the part of the integral under investigation,

$$\int_1^2 [\delta A_x dx + d(A_x \delta x) - dA_x \delta x]$$

$$= \int_1^2 [\delta A_x dx + A_x d\delta x],$$

$$= \int_1^2 [\delta A_x dx + A_x \delta dx],$$

$$= \int_1^2 \delta(A_x dx).$$

Now this expression is the difference of two line integrals, namely

$$\int_{L1}^2 A_x dx - \int_{R1}^2 A_x dx,$$

the one distinguished by the letter L being taken along the left-hand boundary of the d area and the other, marked R, along the right-hand boundary. The difference can be expressed as the line integral

$$\int A_x dx,$$

taken right round the d area in a clockwise sense.

When we take terms involving A_y and A_z into account we find the surface integral $\iint (\mathbf{curl}\ \mathbf{A}, d\mathbf{S})$ over the d area to be equivalent to the line integral $\int (A_x dx + A_y dy + A_z dz)$ taken round it. Otherwise expressed

$$\iint (\mathbf{curl}\ \mathbf{A}, d\mathbf{S}) = \int (\mathbf{A}, d\mathbf{l}),$$

$d\mathbf{l}$ being a vectorial element of length along the boundary of the d area.

Finally the integral $\iint (\mathbf{curl}\ \mathbf{A}, d\mathbf{S})$, when extended over the whole surface, is equivalent to the sum of all the line integrals, $\int (\mathbf{A}, d\mathbf{l})$, taken round all the d areas of which the surface is made up. This sum must be equal to the line integral

taken round the boundary $\alpha\beta\gamma\delta$, since along every other line numerically equal integrals are extended in opposite senses. We thus arrive at the important result known as the **theorem of Stokes,**

$$\oint (\mathbf{A}\,d\mathbf{l}) = \iint (\mathbf{curl}\ \mathbf{A}, d\mathbf{S}) \quad . \quad . \quad . \quad (3\cdot32)$$

The line integral is extended, as the symbol $\oint$ is meant to indicate, round the boundary $\alpha\beta\gamma\delta$ of the surface over which the integral $\iint$ is extended.

If we imagine a screw turning in the sense in which the line integral is taken round the boundary, the vectorial elements of area $d\mathbf{S}$ will be directed to the side towards which the screw is travelling.

§ 4. Fourier's Expansion

A very extensive class of functions can be represented, for a limited range of values of the independent variable, by the sum of a series of trigonometrical terms. If ϕ be the independent variable and $f(\phi)$ the function, we have

$$f(\phi) = A_0 + A_1 \cos \phi + A_2 \cos 2\phi + A_3 \cos 3\phi + \ldots$$
$$+ B_1 \sin \phi + B_2 \sin 2\phi + B_3 \sin 3\phi + \ldots \quad (4)$$

The coefficients are defined by the equations,

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(\tau)d\tau,$$

$$A_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\tau) \cos n\tau \, d\tau, \left.\begin{array}{c}\\ \\ \\ \\ \\ \end{array}\right\} n = +1 \text{ to } +\infty \quad . \quad (4\cdot01)$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\tau) \sin n\tau \, d\tau$$

where n may have all integral values from 1 to ∞. The sum of the series (**4**) will correctly represent the function $f(\phi)$, subject to a qualification given below, for all values of ϕ between $-\pi$ and $+\pi$. The expansion is due to Jean Baptiste Fourier and will be found in his *Théorie Analytique de la Chaleur* published in 1822. Its validity was established by Lejeune Dirichlet in 1837 for all one-valued functions of the type which can be represented graphically. A discussion of the validity of Fourier's expansion (**4**) is beyond the scope of this book; but if we accept

the validity, it is easy to prove that the coefficients are those
defined by (4·01). In evaluating any integral, such as

$$\int_{-\pi}^{+\pi} f(\tau) \cos (n\tau) d\tau,$$

we have simply to make use of the relations :

$$\int_{-\pi}^{+\pi} \cos^2 (n\tau) \, d\tau = \pi,$$

$$\int_{-\pi}^{+\pi} \sin^2 (n\tau) \, d\tau = \pi,$$

$$\int_{-\pi}^{+\pi} \cos (n\tau) \sin (m\tau) \, d\tau = 0, \begin{cases} n = m, \\ \text{or } n \neq m, \end{cases} \quad \cdot \quad \cdot \quad (4\cdot02)$$

$$\int_{-\pi}^{+\pi} \cos (n\tau) \cos (m\tau) \, d\tau = 0, \ n \neq m,$$

$$\int_{-\pi}^{+\pi} \sin (n\tau) \sin (m\tau) \, d\tau = 0, \ n \neq m.$$

In Fig. 4 the abscissæ are the values of the independent
variable, ϕ, and the ordinates those of a function, $f(\phi)$, which
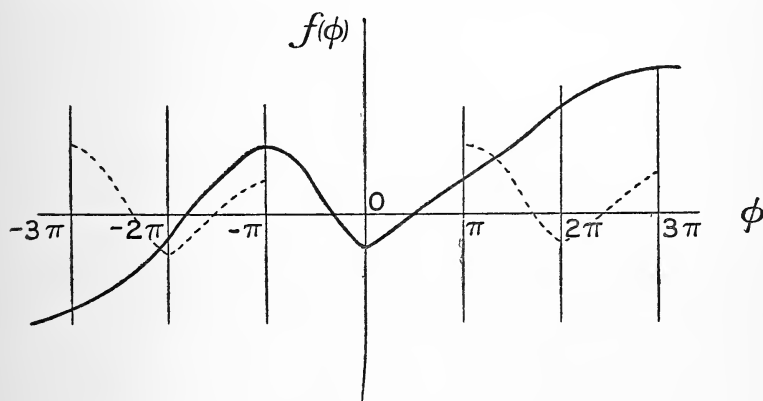
FIG. 4

may be given quite arbitrarily. Whether the function is periodic
or not, the expansion will represent it correctly between $-\pi$

and $+\pi$, with the possible exception of the points $-\pi$ and $+\pi$ themselves. Outside this range of values of the independent variable, it is evident that the arbitrarily given function $f(\phi)$ cannot in general be equivalent to the sum of the series (4), since the periodic character of the cosine and sine terms necessitates that the values of the function in the interval $-\pi$ to $+\pi$ will be reproduced by the expansion in every further interval of 2π, e.g. from π to 3π, or from -3π to $-\pi$. This is indicated in the figure by the broken lines.

We notice that the sum of the series (4) may approach two different limiting values from the two sides of the points $-\pi$ and $+\pi$, and it can be shown that the result obtained by substituting $-\pi$ or $+\pi$ for ϕ is the arithmetic mean of the two limiting values. If we wish to expand an arbitrary function $\psi(x)$ in a series of cosine and sine terms which will be valid for any prescribed range of values of x, we can quite simply reduce the problem to the one we have discussed by introducing a variable

$$\phi = \frac{\pi}{L}x,$$

and we shall arrive at a result which is valid for values of x between $-L$ and $+L$.

When the expressions (4·01) for the coefficients A and B are substituted in (4), Fourier's series takes the form

$$f(\phi) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(\tau)d\tau + \frac{1}{\pi} \sum_{n=+1}^{n=+\infty} \int_{-\pi}^{+\pi} f(\tau) \cos n\tau \cos n\phi \, d\tau$$

$$+ \frac{1}{\pi} \sum_{n=+1}^{n=+\infty} \int_{-\pi}^{+\pi} f(\tau) \sin n\tau \sin n\phi \, d\tau,$$

or

$$f(\phi) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(\tau)d\tau + \frac{1}{\pi} \sum_{n=+1}^{n=+\infty} \int_{-\pi}^{+\pi} f(\tau) \cos n(\tau - \phi) d\tau. \quad (4\cdot03)$$

Since

$$\cos x = \cos (- x)$$

we may evidently write (4·03) in the alternative form

$$f(\phi) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(\tau)d\tau + \frac{1}{\pi} \sum_{n=-1}^{n=-\infty} \int_{-\pi}^{+\pi} f(\tau) \cos n(\tau - \phi) d\tau \quad (4\cdot031)$$

and on adding (4·03) and (4·031) we get

$$2f(\phi) = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(\tau)d\tau + \frac{1}{\pi}\sum_{n=-1\,\text{to}\,-\infty}^{n=+1\,\text{to}\,+\infty}\int_{-\pi}^{+\pi} f(\tau)\cos n(\tau-\phi)d\tau,$$

and hence

$$f(\phi) = \frac{1}{2\pi}\sum_{n=-\infty}^{n=+\infty}\int_{-\pi}^{+\pi} f(\tau)\cos n(\tau-\phi)d\tau . \quad . \quad (4\cdot04)$$

If now in this formula we replace

$$\tau \text{ by } \frac{\sigma}{a},$$

and

$$\phi \text{ by } \frac{\psi}{a},$$

where a is a positive constant, we shall have

$$f(\tau)\cos n(\tau-\phi)d\tau = f\left(\frac{\sigma}{a}\right)\cos\frac{n}{a}(\sigma-\psi)d\sigma.\frac{1}{a}.$$

Write $\dfrac{n}{a} = \lambda$, and suppose a to be very large, approaching ∞ in the limit. Then $\dfrac{1}{a}$ becomes $d\lambda$ and the summation, Σ, with respect to n becomes an integration, $\displaystyle\int$, with respect to λ. Therefore, if we write $F(\sigma)$ for $f\left(\dfrac{\sigma}{a}\right)$ and $F(\psi)$ for $f\left(\dfrac{\psi}{a}\right)$, and observe that the limits for $\sigma = a\tau$ must be $-\infty$ and $+\infty$, we arrive at the interesting result

$$F(\psi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\sigma)\cos \lambda(\sigma-\psi)d\sigma d\lambda . \quad . \quad (4\cdot05)$$

which is known as **Fourier's Theorem.**

The derivation just given is not rigorous, but it shows the connexion between the theorem and Fourier's expansion. It can be proved to be valid for arbitrary functions of the type that can be exhibited graphically, provided the integral $\displaystyle\int_{-\infty}^{+\infty} F(\sigma)d\sigma$ is convergent. The arbitrarily given function $F(\psi)$ may have a finite number of discontinuities, i.e. there may be a finite number of values of ψ at each of which $F(\psi)$ has two limiting values. For these values of ψ the integral in (4·05) gives the mean of

the two limiting values and in this respect is like Fourier's
expansion.

§ 4·1. Examples of Fourier Expansions

When $f(\phi)$ in (4) is equal to ϕ, all the coefficients, A, vanish,
since the value of the function merely changes in sign when
the sign of ϕ is changed.   Therefore

$$f(\phi) = \phi = \sum_{1}^{\infty} B_n \sin n\phi.$$

By (4·01)

$$B_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} \tau \sin n\tau \, d\tau.$$

This gives on integrating by parts,

$$B_n = -\frac{2}{n} \cos n\pi,$$

and therefore

$$\phi = 2 \left\{ \sin \phi - \tfrac{1}{2} \sin 2\phi + \tfrac{1}{3} \sin 3\phi - + \ldots \right\} \quad . \quad (4\cdot1)$$

The sum of the series (4·1) approaches the limit π as ϕ
approaches π from below, and the limit $-\pi$ as ϕ approaches
$-\pi$ from above.   On account of the periodicity of the terms
it will also approach the limit $-\pi$ as ϕ approaches π from
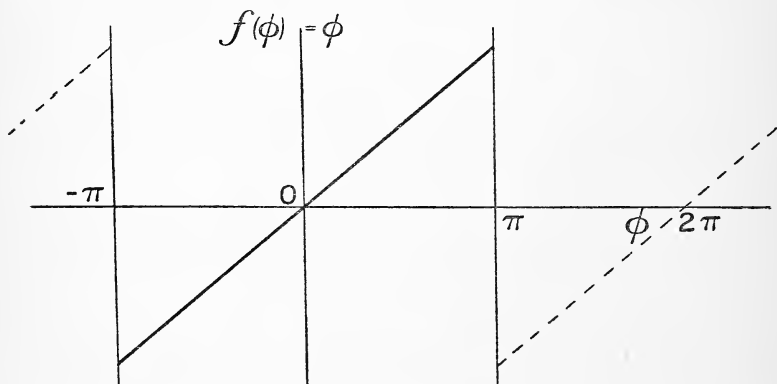
Fig. 4·1

above.   There are therefore two limiting values of the sum of
the series at π and at every point $n\pi$ where n is odd.   When
however we substitute $n\pi$ for ϕ the sum of the series is found
to be zero, which is the arithmetic mean of $-\pi$ and $+\pi$ (Fig. 4·1).

If we wish to represent $\psi(x) = x$ by a trigonometrical series in an arbitrarily given interval, e.g.

$$-L < x < L,$$

we may substitute $\pi\dfrac{x}{L}$ for ϕ in (4·1) thus

$$\frac{\pi x}{L} = 2\left\{\sin \frac{\pi x}{L} - \frac{1}{2}\sin 2\frac{\pi x}{L} + \frac{1}{3}\sin \frac{3\pi x}{L} - + \ldots\right\},$$

and so we get

$$x = \frac{2L}{\pi}\left\{\sin \frac{\pi x}{L} - \frac{1}{2}\sin 2\frac{\pi x}{L} + \frac{1}{3}\sin 3\frac{\pi x}{L} - + \ldots\right\} \quad (4·11)$$

FIG. 4·11

When

$$f(\phi) = \phi, \qquad 0 \gtrless \phi \gtrless \pi,$$

and

$$f(\phi) = -\phi, \quad -\pi \gtrless \phi \gtrless 0,$$
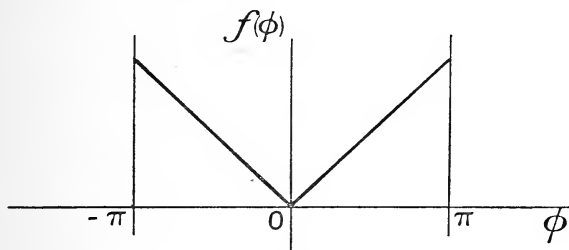
the function is an *even* one and the coefficients B vanish (Fig. 4·11). Therefore

$$f(\phi) = A_0 + \sum_1^\infty A_n \cos n\phi.$$

Equations (4·01) give us

$$A_0 = \frac{1}{2\pi}\left\{\int_{-\pi}^{0} -\tau d\tau + \int_{0}^{\pi} \tau d\tau\right\},$$

or

$$A_0 = \frac{\pi}{2},$$

and

$$A_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\tau \cos n\tau\, d\tau + \int_{0}^{\pi} \tau \cos n\tau\, d\tau\right\}.$$

Integrating by parts, we find

$$A_n = \frac{2}{n^2\pi}(\cos n\pi - 1).$$

Therefore $\qquad A_2 = A_4 = A_6 = \ldots = 0,$

and $\qquad A_1 = -\dfrac{4}{\pi},$

$$A_3 = -\frac{4}{3^2\pi},$$

$$A_5 = -\frac{4}{5^2\pi},$$

and so on. Therefore

$$f(\phi) = \frac{\pi}{2} - \frac{4}{\pi}\left\{\cos\phi + \frac{1}{3^2}\cos 3\phi + \frac{1}{5^2}\cos 5\phi + \ldots\right\} \qquad (4\cdot12)$$
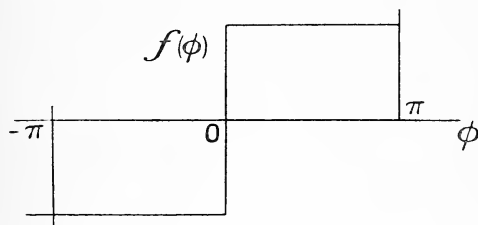
FIG. 4·12

The sums of the series (4·1) and (4·12) are equal when $0 \leq \phi < \pi$. If

$$f(\phi) = -1, \quad \phi < 0,$$
$$f(\phi) = +1, \quad \phi > 0,$$

(Fig. 4·12) we find the expansion

$$f(\phi) = \frac{4}{\pi}\left\{\sin\phi + \frac{1}{3}\sin 3\phi + \frac{1}{5}\sin 5\phi + \ldots\right\} \qquad . \quad . \quad (4\cdot13)$$

If $\qquad\qquad f(\phi) = -1, \quad \begin{cases} \phi < -\dfrac{\pi}{2}, \\ \phi > \dfrac{\pi}{2}, \end{cases}$

and $\qquad\qquad f(\phi) = +1, \quad -\dfrac{\pi}{2} < \phi < \dfrac{\pi}{2},$

(Fig. 4·13) we obtain

$$f(\phi) = \frac{4}{\pi}\left\{\cos\phi - \frac{1}{3}\cos 3\phi + \frac{1}{5}\cos 5\phi - + \ldots\right\} \qquad (4\cdot14)$$
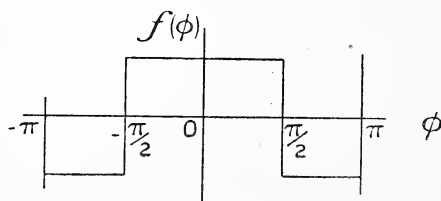
FIG. 4·13

As a further example take the case, illustrated in Fig. 4·14, where
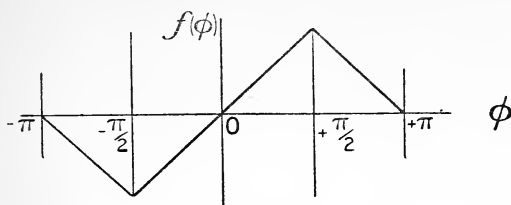
FIG. 4·14

$$f(\phi) = \phi, \qquad\qquad -\frac{\pi}{2} \lessgtr \phi \lessgtr \frac{\pi}{2},$$

$$f(\phi) = \pi - \phi, \qquad\qquad \frac{\pi}{2} \lessgtr \phi \lessgtr \pi,$$

$$f(\phi) = -(\pi + \phi), \qquad -\pi \lessgtr \phi \lessgtr \frac{\pi}{2}.$$

The function is an odd one and hence the coefficients A are all zero, while

$$B_n = \frac{4}{n^2\pi} \sin n\frac{\pi}{2},$$

so that the appropriate expansion is

$$f(\phi) = \frac{4}{\pi}\left\{ \sin\phi - \frac{1}{3^2}\sin 3\phi + \frac{1}{5^2}\sin 5\phi - + \ldots \right\} \quad (4·15)$$

As a concluding example we may take the function

$$f(\phi) = -\sin\phi, \quad -\pi \leq \phi \leq 0,$$
$$f(\phi) = +\sin\phi, \quad 0 \leq \phi \leq \pi,$$

(see Fig. 4·15), for which we find the expansion

$$f(\phi) = \frac{2}{\pi} - \frac{4}{\pi}\left\{ \frac{\cos 2\phi}{1·3} + \frac{\cos 4\phi}{3·5} + \frac{\cos 6\phi}{5·7} + \ldots \right\} \quad (4·16)$$
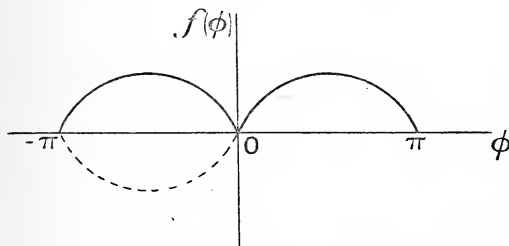
FIG. 4·15

In all the examples where points of discontinuity occur, e.g. at $\phi = 0$ in (4·13) and at $\phi = -\frac{\pi}{2}$ and $\phi = +\frac{\pi}{2}$ in (4·14), it

may be verified that at such points the sum of the series, which is of course a one-valued thing, is equal to the mean of the two limiting values of $f(\phi)$ at the point in question.

We obtain interesting verifications of the formulæ given above when we substitute special numerical values for ϕ. For example, if we substitute the value $\dfrac{\pi}{2}$ for ϕ in (**4·1**) or in (**4·13**), or if we substitute the value 0 in (**4·14**) we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \ldots$$

If we put $\phi = 0$ in (**4·12**), or $\phi = \dfrac{\pi}{2}$ in (**4·15**) we obtain

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$

Both of these formulæ are well known, and can easily be arrived at in other ways.

It is possible, and often convenient, to give the expansion (**4**) the form

$$f(\phi) = \sum_{\nu=-\infty}^{\nu=+\infty} a_\nu e^{i\nu\phi} \qquad \ldots \quad \textbf{(4·17)}$$

This is effected by the substitutions

$$\cos n\phi = \frac{e^{in\phi} + e^{-in\phi}}{2},$$

$$\sin n\phi = \frac{e^{in\phi} - e^{-in\phi}}{2i},$$

where i is the usual abbreviation for $\sqrt{-1}$. The series (**4·1**), for example, when expressed in this way, becomes

$$\phi = - i\, e^{i\phi} + \frac{i}{2} e^{i2\phi} - \frac{i}{3} e^{i3\phi} + - \ldots$$

$$+ i\, e^{-i\phi} - \frac{i}{2} e^{-i2\phi} + \frac{i}{3} e^{-i3\phi} - + \ldots$$

§ 4·2. Orthogonal Functions

The most important property of the trigonometrical functions in a Fourier series is that which finds its expression in equations (**4·02**). An aggregate of functions like $\cos n\tau$ and $\sin n\tau$ with this property is called a **system of orthogonal functions,** because of the analogy between the equations (**4·02**) and the equations (**2·23**) and (**2·24**) which give the relations between

the direction cosines of mutually perpendicular lines. The analogy becomes more obvious if we adopt the functions

$$\frac{\cos (n\tau)}{\sqrt{\pi}} \quad \text{and} \quad \frac{\sin (n\tau)}{\sqrt{\pi}}$$

instead of $\cos (n\tau)$ and $\sin (n\tau)$, for then the non-vanishing integrals of (4·02) are

$$\int_{-\pi}^{+\pi} \left\{\frac{\cos (n\tau)}{\sqrt{\pi}}\right\}^2 d\tau = 1$$

$$\int_{-\pi}^{+\pi} \left\{\frac{\sin (n\tau)}{\sqrt{\pi}}\right\}^2 d\tau = 1.$$

When the non-vanishing integrals are thus modified so as to have the value unity, the orthogonal functions are said to be **normed.** For the interval $-\pi$ to $+\pi$ the normed trigonometrical orthogonal functions are

$$\frac{\cos (n\phi)}{\sqrt{\pi}} \quad \text{and} \quad \frac{\sin (n\phi)}{\sqrt{\pi}}, \quad n = 1, 2, \ldots \infty$$

and to these we may add the constant $\dfrac{1}{\sqrt{2\pi}}$, since

$$\int_{-\pi}^{+\pi} \left\{\frac{1}{\sqrt{2\pi}}\right\}^2 d\tau = 1.$$

In terms of the normed functions, the Fourier expansion becomes

$$f(\phi) = a_0\left(\frac{1}{\sqrt{2\pi}}\right) + a_1 \frac{\cos \phi}{\sqrt{\pi}} + a_2 \frac{\cos 2\phi}{\sqrt{\pi}} + \ldots$$

$$+ \beta_1 \frac{\sin \phi}{\sqrt{\pi}} + \beta_2 \frac{\sin 2\phi}{\sqrt{\pi}} + \ldots \qquad (4·2)$$

the coefficients being given by

$$a_0 = \int_{-\pi}^{+\pi} f(\tau)\left(\frac{1}{\sqrt{2\pi}}\right) d\tau,$$

$$a_n = \int_{-\pi}^{+\pi} f(\tau)\left\{\frac{\cos (n\tau)}{\sqrt{\pi}}\right\} d\tau, \quad n = 1, 2, 3 \ldots \infty.$$

$$\beta_n = \int_{-\pi}^{+\pi} f(\tau)\left\{\frac{\sin (n\tau)}{\sqrt{\pi}}\right\} d\tau, \quad n = 1, 2, 3 \ldots \infty.$$

Let us now introduce the notation,

$$E_0(\phi) = \frac{1}{\sqrt{2\pi}}, \qquad E_1(\phi) = \frac{\sin \phi}{\sqrt{\pi}},$$

$$E_2(\phi) = \frac{\cos \phi}{\sqrt{\pi}}, \qquad E_3(\phi) = \frac{\sin (2\phi)}{\sqrt{\pi}},$$

$$E_4(\phi) = \frac{\cos (2\phi)}{\sqrt{\pi}}, \qquad E_5(\phi) = \frac{\sin (3\phi)}{\sqrt{\pi}},$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$E_{2m}(\phi) = \frac{\cos (m\phi)}{\sqrt{\pi}}, \qquad E_{(2m+1)} = \frac{\sin (m+1)\phi}{\sqrt{\pi}}.$$

The Fourier expansion now becomes

$$f(\phi) = \overset{\infty}{\underset{0}{\Sigma}} c_n E_n(\phi) \qquad . \quad . \quad . \quad . \quad (4\cdot21)$$

and since the integral relations $(4\cdot02)$ now have the form

$$\int_{-\pi}^{+\pi} E_n(\tau) E_m(\tau) d\tau = 1, \quad n = m,$$

$$= 0, \quad n \neq m \quad . \quad . \quad . \quad (4\cdot22)$$

the coefficients c_n are given by

$$c_n = \int_{-\pi}^{+\pi} f(\tau) E_n(\tau) d\tau, \quad n = 0, 1, 2 \ldots \infty \quad . \quad (4\cdot23)$$

Systems of orthogonal functions are of course not limited to trigonometrical functions. A system of orthogonal functions $E_n(\phi)$ by means of which an arbitrary function $f(\phi)$ can be expanded in the form $(4\cdot21)$ is called a **complete system** of orthogonal functions.

CHAPTER III

INTRODUCTION TO DYNAMICS

§ 5. FORCE, MASS, NEWTON'S LAWS

THE notion of force has its origin in the feeling of muscular effort. Quantitative estimates based immediately on the feeling or sensation of effort are, however, too rough and uncertain to serve any purposes where precision and consistency are demanded. Consequently force, like all other physical quantities, is measured by devices which entirely eliminate any dependence on the intensity of a sensation. Typical of such devices is a spring (as in a spring balance, for example). Instead of the uncertain comparison of two weights by feeling how big are the muscular efforts exerted in supporting them, it is better to use the extensions they produce in a spring as measures of their weights. The procedure in measuring temperatures is quite analogous. The sensation of warmth or hotness plays no part whatever in such measurements. Let us examine more closely the measurement of force by the extension of a spring. To fix our ideas we may continue to keep the spring balance in mind. The upper end, A, of the spring is attached to a support, which for our purposes may be supposed to be rigidly fixed. A heavy body is suspended at the lower end, B. The extension can be represented completely by a vertical line (XY in Fig. 5) the length of which is made equal to it. The upward and downward directions in XY have equal claims on our attention, and we shall say that a force is exerted in an upward direction on the body suspended at B, and that a force is exerted in a downward direction on the support at A. *Both forces* are measured by the extension, XY, of the spring and are therefore numerically equal to one another. This is the (so-called) **law of action and reaction,** which here emerges as a necessary consequence of the measuring device and associated definitions. It is implied, in what has been said, that the unit force is the force associated with the unit extension, and it follows that different units of length will, in

FIG. 5

general, have to be used with different springs. There is no difficulty in deciding when the extensions produced in a number of springs are all associated with the same force ; but the following circumstance has to be considered : though different springs may agree with one another approximately, or even very closely, when used to measure forces extending over a certain limited range, and though the intervals on the evenly-divided scales attached to them may have been chosen so that all the springs are in precise agreement in the case of one particular force (the adopted unit) ; they will nevertheless be found to disagree to some extent when measuring other forces. In fact, when springs are used in the way described, each measures forces according to a scale of its own.   There is a close parallel to this in the measurement of temperature by thermometers of different types.   We find it necessary to define a **force scale** independent of the peculiarities of the particular measuring device—something precisely similar is done in the measurement of temperature. We might do this by adjusting a number of weights so that they all produce the same extension in one particular spring, when hung from it separately.   We should then be able to calibrate any spring by suspending the weights from it, one, two or more at a time.   The following observational facts point out another way of defining a force scale (which will, in fact, amount to the same thing in the end) : In the first place, if the point of support, A, referred to above, be caused to ascend with a uniform speed, the extension of the spring (after the initial oscillations have been damped out) will not be altered.   This means that no force is needed to maintain the *constant velocity* of the body at B.   In the next place, if A ascends with a *constant acceleration*, the extension of the spring will be *increased* by a definite amount, proportional (approximately) to the acceleration.   In other words, the upward acceleration of the body suspended at B is approximately proportional to the resultant force (as measured by the spring) to which the acceleration is due.   In consequence of these facts it is possible, and many reasons make it desirable, to *define* the force scale by the statement

$$\mathbf{F} = m\mathbf{a} \qquad . \quad . \quad . \quad . \quad . \quad (5)$$

where $\mathbf{F}$ is the resultant force, $\mathbf{a}$ is the acceleration, and m is a constant characteristic of the particular body, and called its **mass**. If we make m unity for some arbitrarily chosen body, we shall thereby fix the unit of force at the same time.   It will in fact be the force which causes it to move with the unit acceleration.   In the case of bodies made entirely of the same material, e.g. brass, the mass is found to be very nearly propor-

tional to the volume. This is the justification of the rather
imperfect definition of the mass of a body as the quantity of
material in it.

Representing the velocity of the body by **v**, equation (5) may
be written

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt},$$

or

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}),$$

or finally

$$\mathbf{F} = \frac{d\mathbf{M}}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}01)$$

where

$$\mathbf{M} = m\mathbf{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}011)$$

is called the **momentum** of the body. Equation (5·01)
embodies in a single statement Newton's first and second laws
of motion ; the law of action and reaction being Newton's
third law. In applying these laws generally, we must regard
the body as very small in its dimensions, i.e. as a **particle**, in
order to avoid the difficulty which would appear if the velocities
or accelerations of its parts differed from one another ; and in
dealing with classical dynamics we shall assume, as Newton
appears to have done, that the mutual forces, exerted by two
particles on one another, are directed along the straight line
joining them.

The unit of mass adopted for scientific purposes is the **gram,**
i.e. the mass of a cubic centimetre of water at the temperature
of its maximum density ; the unit of length is the **centimetre,**
and the unit of time the **mean solar second.** With these funda-
mental units, the unit of force fixed by (5) or (5·01), i.e. the force
causing the unit rate of change of momentum, is called the **dyne.**

It is an experimental fact of great importance for the science
of physics that unsupported bodies at the same place, i.e. bodies
which have been projected and are falling freely, have the *same
downward acceleration*. This is usually represented by the letter
g, and is equal to 980·6 cm. sec.$^{-2}$ in latitude 45°, and varies from
978 cm. sec.$^{-2}$ at the equator to 983·4 cm. sec.$^{-2}$ at the poles.
By ' freely falling ' body is to be understood of course one which
is not subject to the resistance of the air or any other sort of
interference. There is therefore a downward force acting on the
body, equal to mg. This is its **weight.** At the same place,
therefore, the weights of bodies are proportional to their masses ;
but whereas the mass of a body is a constant characteristic of it,[1]
and independent of its geographical position, its weight will vary

[1] This statement will be modified when the theory of relativity is
described.

with the latitude in consequence of the variation of g. The acceleration, g, is the weight or gravitational force *per unit mass* and we shall term it shortly the **intensity of gravity.** This is, for several reasons, preferable to the ambiguous term ' acceleration due to gravity '.

§ 5·1. WORK AND ENERGY

The scalar product (**Fdl**), or $F_x dx + F_y dy + F_z dz$, where $\mathbf{F} \equiv (F_x, F_y, F_z)$ is any force and $\mathbf{dl} \equiv (dx, dy, dz)$ is a small displacement of its point of application, is called the **work** done by the force during the displacement **dl.** And when the point of application of the force travels from any point A along some path ABC to another point C (Fig. 5·1), the work done may be represented by
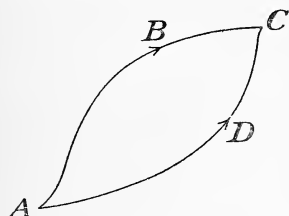
FIG. 5·1

$$\int_{ABC} (\mathbf{Fdl}).$$

It may happen (and this is a very important case) that the work done is independent of the path. Starting from some fixed point A, the work done will depend *only on the position of* C. If we represent the work by W, we have $W = $ function (x,y,z), where x, y, z are the co-ordinates of C. And if we take some neighbouring point C′ with co-ordinates $(x + dx, y + dy, z + dz)$ the work done will be $W + dW$, where

$$dW = \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz \qquad . \quad . \quad (5·1)$$

so that

$$F_x = \frac{\partial W}{\partial x}, \; F_y = \frac{\partial W}{\partial y}, \; F_z = \frac{\partial W}{\partial z}.$$

We may therefore say that, when the work done is independent of the path, the force is the gradient of a scalar quantity. Conversely, when the force is the gradient of a scalar quantity at all points in a certain region, the work done between two points A and C in the region is independent of the path, since the curl of a gradient is zero (§ 2·4) and therefore

$$\iint \left(\mathbf{curl\,F, dS} \right)$$

and consequently, by the theorem of Stokes, the integral

$$\oint (\mathbf{Fdl})$$

taken round any closed loop ABCDA in the region will be zero.

If the work done depends on the path, then the integral $\oint (Fdl)$ round a closed loop such as ABCDA will, in general, differ from zero ; and since

$$\oint (\mathbf{Fdl}) = \iint (\mathbf{curl\ F,\ dS}),$$

we see that **curl F** cannot be everywhere zero. Therefore **F** cannot be represented by a gradient everywhere. When dW is expressed by formula (**5·1**) it is called a **complete** or **perfect differential**.

If we adopt the centimetre, gram and second as fundamental units for length, mass and time respectively, the unit of work derived from them is termed the **dyne-centimetre,** or more usually the **erg**. It is the work done by a force of one dyne when the point where it is applied moves one centimetre in the direction of the force.

Let us consider the motion of a particle under the influence of a force $\mathbf{F} \equiv (F_x, F_y, F_z)$. By equation (**5**) (Newton's second law) we have

$$m\frac{d^2x}{dt^2} = F_x,$$

$$m\frac{d^2y}{dt^2} = F_y \quad . \quad . \quad . \quad . \quad . \quad (\textbf{5·11})$$

$$m\frac{d^2z}{dt^2} = F_z.$$

Multiplying these equations by $\dfrac{dx}{dt}, \dfrac{dy}{dt}$ and $\dfrac{dz}{dt}$ respectively we get

$$m\frac{dx}{dt}\frac{d^2x}{dt^2} = \frac{m}{2}\frac{d}{dt}\left\{\left(\frac{dx}{dt}\right)^2\right\} = F_x\frac{dx}{dt},$$

$$m\frac{dy}{dt}\frac{d^2y}{dt^2} = \frac{m}{2}\frac{d}{dt}\left\{\left(\frac{dy}{dt}\right)^2\right\} = F_y\frac{dy}{dt},$$

$$m\frac{dz}{dt}\frac{d^2z}{dt^2} = \frac{m}{2}\frac{d}{dt}\left\{\left(\frac{dz}{dt}\right)^2\right\} = F_z\frac{dz}{dt};$$

and, on adding, we have

$$\frac{d}{dt}\left\{\frac{m}{2}\mathbf{v}^2\right\} = F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dt}$$

where **v** is the velocity of the particle. Therefore

$$\frac{m}{2}\mathbf{v}^2 - \frac{m}{2}\mathbf{v}_0{}^2 = \int \{F_x dx + F_y dy + F_z dz)$$

or $\qquad \dfrac{m}{2}\mathbf{v}^2 - \dfrac{m}{2}\mathbf{v}_0{}^2 = \displaystyle\int (\mathbf{Fdl}) \ . \quad . \quad . \quad . \quad . \quad . \quad (\textbf{5·12})$

In this equation $\mathbf{v}$ and $\mathbf{v}_0$ mean the velocities of the particle at the end and at the beginning of the path over which the integration is extended.

When the force is the gradient of a scalar W the integral on the right of (5·12) has the form

$$\int \left\{ \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz \right\}$$
$$= W - W_0,$$

and therefore equation (5·12) becomes

$$\frac{m\mathbf{v}^2}{2} - W = \frac{m\mathbf{v}_0{}^2}{2} - W_0 . \quad . \quad . \quad . \quad (5\cdot121)$$

If we replace W by $-V$ and use the letter T for $\dfrac{m\mathbf{v}^2}{2}$, we have

$$T + V = T_0 + V_0 \quad . \quad . \quad . \quad . \quad (5\cdot122)$$

The quantity $T + V$ remains constant. This quantity is called the **energy** of the particle, T being its **kinetic** energy, a function of its mass and velocity, and V the **potential** energy, a function of its position. Equation (5·122) affirms that the energy of the particle remains constant.

The conception of energy will be developed more fully in subsequent chapters ; for the present we are concerned only with the two kinds of energy which have just been described, and it will be noted that energy is measured in terms of the same unit as work. We shall normally regard the centimetre, gram, and second as our fundamental units—the question as to whether three fundamental units will suffice may be deferred till later— but it is obvious that the foregoing formulae (5·11) *et seq.* are quite independent of the particular choice we make of fundamental units.

§ 5·2. Centre of Mass

Imagine a number of particles, the masses of which are $m_1, m_2, m_3, \ldots m_s$, and their co-ordinates $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), \ldots (x_s, y_s, z_s)$ respectively. The **centre of mass** (x, y, z) of the system of particles is defined by

$$Mx = \Sigma m_s x_s,$$
$$My = \Sigma m_s y_s . \quad . \quad . \quad . \quad . \quad (5\cdot2)$$
$$Mz = \Sigma m_s z_s,$$

where $\qquad M = \Sigma m_s,$

the summation being extended over all the particles of the

system.   Let F_x be the sum of all the X components of the forces exerted on the particles of the system ;   we shall have

$$\Sigma m_s \frac{d^2x_s}{dt^2} = F_x,$$

and two corresponding equations for F_y and F_z ;   or by (5·2)

$$M\frac{d^2x}{dt^2} = F_x,$$

$$M\frac{d^2y}{dt^2} = F_y \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot21)$$

$$M\frac{d^2z}{dt^2} = F_z.$$

Therefore the motion of the centre of mass is the same as it would be if all the masses were concentrated in it and all the forces applied there.   In equations (5·21) we may regard **F** as the resultant of all the forces of *external* origin, since by Newton's third law, those of *internal* origin will annul one another. An important case is that in which the system is free from external forces.   In this case we can infer from (5·21) that the centre of mass will move with a constant velocity—which may of course be zero.   It is immaterial whether the particles constitute a rigid body or not.

§ 5·3.   PATH OF PROJECTILE

Let a particle of mass m be projected from the origin of rectangular co-ordinates with a given initial velocity, and suppose the X axis directed vertically upwards.   If g is the intensity of gravity, i.e. if g is the weight of the unit mass, the equations of motion of the particle are :

$$m\frac{d^2x}{dt^2} = -mg,$$

$$m\frac{d^2y}{dt^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5\cdot3)$$

$$m\frac{d^2z}{dt^2} = 0.$$

The two latter equations (5·3) give us

$$y = a_1 t + \beta_1,$$
$$z = a_2 t + \beta_2 \quad . \quad . \quad . \quad . \quad . \quad (5\cdot31)$$

where $a_1, \beta_1, a_2, \beta_2$, are constants.   If we eliminate t from (5·31) we get the equation

$$z = a_2 \frac{(y - \beta_1)}{a_1} + \beta_2$$

or

$$\qquad a_1 z = a_2 y + a_1 \beta_2 - a_2 \beta_1 \quad . \quad . \quad . \quad . \quad (5\cdot311)$$

which is the equation of a vertical plane. The path of the particle therefore lies in a vertical plane, and it is convenient to place our axes of co-ordinates so that this vertical plane is the XY plane, and the Z co-ordinate of the particle is permanently equal to zero. The equations of motion are now

$$\frac{d^2x}{dt^2} = -g$$

$$\frac{d^2y}{dt^2} = 0.$$

Therefore

$$\begin{rcases} x = -\tfrac{1}{2}gt^2 + At + B \\ y = \qquad\quad at + \beta \end{rcases} \quad . \quad . \quad . \ (5\cdot312)$$

If we eliminate t from these equations we obtain the equation of the trajectory of the particle,

$$x = -\tfrac{1}{2}g\frac{(y-\beta)^2}{a^2} + A\Big(\frac{y-\beta}{a}\Big) + B \quad . \quad . \ (5\cdot32)$$

The constants m and g, which appear already in the differential equations (**5·3**) before any steps have been taken to integrate them, we shall term **inherent** constants. Such constants are characteristic of the system to which they belong and are not in any way at our disposal. It is otherwise with the constants of **integration** A, B, a and β. If the particle is projected from the origin at the time $t = 0$ with a velocity V and in a direction making an angle θ with the horizontal axis, Y, we have from (**5·312**)

$$B = 0,$$
$$\beta = 0,$$

and since

$$\frac{dx}{dt} = -gt + A,$$

$$\frac{dy}{dt} = a,$$

therefore

$$V \sin \theta = A,$$
$$V \cos \theta = a.$$

Substituting in (**5·32**), we have

$$x = -\tfrac{1}{2}g\frac{y^2}{V^2 \cos^2 \theta} + \frac{V \sin \theta}{V \cos \theta} y$$

or $\quad 2x . V^2 \cos^2 \theta = -gy^2 + 2V^2 \sin \theta . \cos \theta . y \quad . \quad (5\cdot321)$

If in (**5·321**) we put $x = 0$, we find

$$y = 0$$

or

$$y = \frac{V^2 \sin 2\theta}{g},$$

the latter of these values of y representing the horizontal distance travelled by the projectile between two instants when it is in the plane $x = 0$.

§ 5·4. MOTION OF A PARTICLE UNDER THE INFLUENCE OF A CENTRAL ATTRACTING OR REPELLING FORCE

Let the point, towards or away from which the force on the particle is directed, be the origin, O, of rectangular axes of co-ordinates, and let

$$\mathbf{r} \equiv (x, y, z)$$

be the displacement of the particle from the origin. Let the absolute value of the force be $\mathbf{F}$. We have for the equations of motion,

$$m\frac{d^2x}{dt^2} = \mathbf{F} \cdot \frac{x}{r},$$

$$m\frac{d^2y}{dt^2} = \mathbf{F} \cdot \frac{y}{r} \qquad . \quad . \quad (5\cdot4)$$

$$m\frac{d^2z}{dt^2} = \mathbf{F} \cdot \frac{z}{r},$$
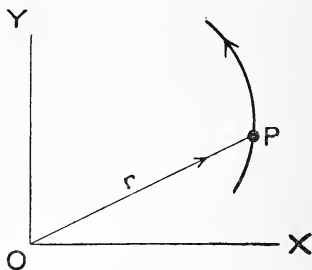
since $\mathbf{F}$ is directed along the radius vector $\mathbf{r}$. Multiply the first two of these equations by y and x respectively and subtract. We thus obtain

$$mx\frac{d^2y}{dt^2} - my\frac{d^2x}{dt^2} = 0,$$

or

$$m\frac{d}{dt}\left(x\frac{dy}{dt}\right) - m\frac{d}{dt}\left(y\frac{dx}{dt}\right) = 0.$$

Consequently

$$mx\frac{dy}{dt} - my\frac{dx}{dt} = \Omega_z,$$

where Ω_z is a constant.

Similarly

$$my\frac{dz}{dt} - mz\frac{dy}{dt} = \Omega_x, \qquad . \quad . \quad . \quad . \quad (5\cdot41)$$

$$mz\frac{dx}{dt} - mx\frac{dz}{dt} = \Omega_y.$$

Evidently Ω_x, Ω_y, Ω_z are the components of a vector. In fact

$$\boldsymbol{\Omega} = m[\mathbf{rv}] \quad . \quad . \quad . \quad . \quad (5\cdot411)$$

(§ 2·1) where $\mathbf{v}$ is the velocity of the particle. The constant $\boldsymbol{\Omega}$ is called the **angular momentum** of the particle. Since

FIG. 5·4

the vector product $[\mathbf{rv}]$ has the absolute value $\mathbf{rv} \sin \theta$ (Fig. 5·41), it must be equal to twice the area swept out per unit time by the radius vector $\mathbf{r}$.   For if in Fig. 5·41 the points 1 and 2 represent two neighbouring positions of the particle $\mathbf{dl}$ apart, the corresponding area swept out by the radius vector is $\frac{1}{2}.\mathbf{r} \sin \theta.\mathbf{dl}$, and dividing by dt, the time taken by the particle to travel $\mathbf{dl}$, we see that the area swept out per unit time is in fact $\frac{1}{2}\mathbf{r} \sin \theta \mathbf{v}$.

The *angle* swept out by the radius vector while the particle travels the distance $\mathbf{dl}$ is

$$\frac{\mathbf{dl} \sin \theta}{\mathbf{r}},$$

and consequently the angular velocity of the particle is

$$\omega = \frac{\mathbf{v} \sin \theta}{\mathbf{r}}.$$

We may therefore write

$$\Omega = m\mathbf{r}^2 . \omega \qquad . \quad (5\cdot412)$$

FIG. 5·41

We may summarize as follows :—**When a particle moves under the influence of a force directed towards or away from a fixed point its angular momentum remains constant.**

Since Ω is equal to the vector product of $\mathbf{r}$ and $\mathbf{v}$ multiplied by the invariant factor m, it must be at right angles to the directions $\mathbf{r}$ and $\mathbf{v}$ (see § 2·1).   Therefore the scalar product $(\Omega \mathbf{r})$ is equal to zero, that is

$$\Omega_x x + \Omega_y y + \Omega_z z = 0 \quad . \quad . \quad . \quad . \quad (5\cdot42)$$

This is the equation of a plane passing through the origin, i.e. through the attracting centre.

§ 5·43.   Angular Momentum of a System of Particles free from External Forces

We may write the equations of motion of any one of the particles, which we shall distinguish by the subscript, s, in the usual way :

$$m_s \frac{d^2 x_s}{dt^2} = F_{sx},$$

$$m_s \frac{d^2 y_s}{dt^2} = F_{sy},$$

$$m_s \frac{d^2 z}{dt^2} = F_{sz}.$$

By a procedure similar to that used to deduce equations (5·41) we get :

$$m_s\frac{d}{dt}\left\{y_s\frac{dz_s}{dt} - z_s\frac{dy_s}{dt}\right\} = y_sF_{sz} - z_sF_{sy},$$

$$m_s\frac{d}{dt}\left\{z_s\frac{dx_s}{dt} - x_s\frac{dz_s}{dt}\right\} = z_sF_{sx} - x_sF_{sz}, \qquad . \quad (5·43)$$

$$m_s\frac{d}{dt}\left\{x_s\frac{dy_s}{dt} - y_s\frac{dx_s}{dt}\right\} = x_sF_{sy} - y_sF_{sx},$$

which may be expressed more briefly in the following form :

$$\frac{d}{dt}\Omega_s = [\mathbf{r}_s\mathbf{F}_s] \quad . \quad . \quad . \quad . \quad (5·431)$$

The vector product $[\mathbf{r}_s\mathbf{F}_s]$ is called the **moment** of the force $\mathbf{F}_s$ with respect to the origin. If we add all the equations (5·431) for the whole system of particles, we have

$$\frac{d}{dt}\Sigma_s\Omega_s = \overline{\Sigma}_s[\mathbf{r}_s\mathbf{F}_s] \quad . \quad . \quad . \quad . \quad (5·432)$$

the summation being a *vectorial* one; i.e.

$$\Sigma_s\Omega_s \equiv \left(\Sigma_s\Omega_{sx}, \ \Sigma_s\Omega_{sy}, \ \Sigma_s\Omega_{sz}\right),$$

$$\Sigma_s[\mathbf{r}_s\mathbf{F}_s] \equiv \left(\Sigma_s(y_sF_{sz} - z_sF_{sy}), \ \Sigma_s(z_sF_{sx} - x_sF_{sz}), \ \Sigma_s(x_sF_{sy} - y_sF_{sx})\right)$$

Equation (5·432) states that the rate of increase of the resultant angular momentum about the origin is equal to the resultant of the moments of the forces with respect to the origin. Now there are by hypothesis no forces of external origin, and if we suppose that the mutual action between any two particles is directed along the straight line joining them, the right-hand side of (5·432) can be shown to be zero. For the moment of a force is numerically equal to the product of its absolute value and the perpendicular distance from the origin to its line of action. Therefore the resultant of the moments due to the mutual action of any two particles on one another must vanish, since the forces are equal and opposite and the perpendicular distance mentioned above is the same for both.

We conclude therefore that **the resultant angular momentum of a system of particles remains constant in magnitude and direction provided the only forces are those due to the mutual action of the particles on one another and that the force exerted by any particle on another is directed along the straight line joining them.**

§ 5·5. PLANETARY MOTION

We shall now study the motion of a particle under the influence of a central force varying inversely as the square of the distance of the particle from a fixed point, which we shall take as the origin of a system of rectangular co-ordinates. Since, as we have already proved (§ 5·4), the particle moves in a plane, we shall place our co-ordinates so that this plane is the XY plane. The Z co-ordinate of the particle will remain constant and equal to zero. We now introduce polar co-ordinates, r and θ, defined by

$$x = r \cos \theta,$$
$$y = r \sin \theta,$$

The kinetic energy of the particle is easily calculated. Since if $\mathbf{dl}$ is an element of the path of the particle,

$$
\begin{aligned}
dl^2 &= dx^2 + dy^2 \\
&= (dr . \cos \theta - r \sin \theta . d\theta)^2 + (dr . \sin \theta + r \cos \theta . d\theta)^2 \\
&= dr^2 + r^2 d\theta^2
\end{aligned}
$$

Therefore the kinetic energy

$$\tfrac{1}{2}m\left(\frac{dl}{dt}\right)^2 = \tfrac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \tfrac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2 \quad . \quad . \quad \textbf{(5·5)}$$

We may eliminate the angular velocity, $\dfrac{d\theta}{dt}$, by making use of

$$\Omega = mr^2\left(\frac{d\theta}{dt}\right) \quad . \quad . \quad . \quad . \quad \textbf{(5·501)}$$

The kinetic energy may thus be expressed in the form

$$\tfrac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{\Omega^2}{2mr^2}.$$

The central force is expressed by

$$\frac{B}{r^2}$$

where B is a constant, positive or negative, according as it is a repulsion or attraction.

The X component of the force will therefore be

$$F_x = \frac{Bx}{r^3}$$

or

$$F_x = -\frac{\partial}{\partial x}\left(\frac{B}{r}\right).$$

Therefore $\dfrac{B}{r}$, plus a constant, is the potential energy of the

particle (see § 5·1).   It is convenient to fix the arbitrary constant in such a way as to make the potential energy zero when the particle is infinitely distant from the centre of force.   Representing the sum of the kinetic and potential energies by E, we have

$$\tfrac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{\Omega^2}{2mr^2} + \frac{B}{r} = E \quad . \quad . \quad . \quad (5\text{·}51)$$

In this equation Ω and E are constants of integration (§§ 5·1 and 5·4).   To get the equation of the path or orbit of the particle we eliminate dt in (5·51) by means of the equation of angular momentum (5·501) which may be expressed in the form,

$$dt = \frac{mr^2}{\Omega}d\theta.$$

Equation (5·51) now becomes

$$\frac{\Omega^2}{2mr^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{\Omega^2}{2mr^2} + \frac{B}{r} = E \quad . \quad . \quad . \quad (5\text{·}511)$$

We now introduce a new variable u defined by

$$u = \frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad (5\text{·}512)$$

so that

$$dr = -\frac{du}{u^2} \quad . \quad . \quad . \quad . \quad . \quad (5\text{·}513)$$

Substituting in equation (5·511) we get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 + \frac{2mB}{\Omega^2}u = \frac{2mE}{\Omega^2} \quad . \quad . \quad (5\text{·}514)$$

Differentiating this equation with respect to θ and dividing out the common factor $2\dfrac{du}{d\theta}$ we obtain

$$\frac{d^2u}{d\theta^2} + u = -\frac{mB}{\Omega^2} \quad . \quad . \quad . \quad . \quad (5\text{·}52)$$

The general solution of (5·52) is

$$u = -\frac{mB}{\Omega^2} + R\cos(\theta - \eta) \quad . \quad . \quad . \quad (5\text{·}53)$$

where R and η are constants of integration.   This equation may be expressed in the form :

$$r = \frac{-\dfrac{\Omega^2}{mB}}{1 + \varepsilon\cos(\theta - \eta)} \quad . \quad . \quad . \quad (5\text{·}531)$$

where ε and η are constants of integration.   The latter evidently depends on the fixed direction, that of the x axis, from which

the angle θ is measured and it is convenient to choose our system of reference so that $\eta = 0$. There is no loss of generality in taking ε to be positive only, since the effect of changing the sign of ε is just the same as that of rotating the fixed direction, from which θ is measured, through the angle π. Equation (5·531) represents a conic section; hyperbola, parabola or ellipse. If the central force is one of repulsion, B is positive and therefore the numerator of (5·531) is negative; and since only positive values of r have a physical significance, it follows that the denominator must also be negative. This is only possible when ε is greater than unity. The orbit of the particle is therefore one branch of a hyperbola (represented by the full line in Fig. 5·5).
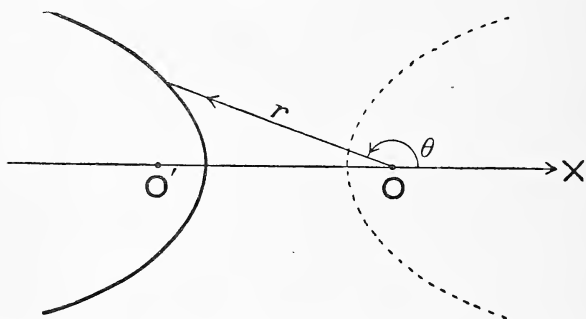
Fig. 5·5

The centre of force is the focus O. It will be seen that the path of the particle goes round the *other* focus, O'. The asymptotes make angles θ_0 with the axis of reference (the X axis) determined by the equation

$$\cos \theta_0 = -\frac{1}{\varepsilon}.$$

If the central force is one of attraction (B negative), the numerator of (5·531) is now *positive* and consequently the denominator must be positive too, in order to furnish physically significant values of r. It is obvious that ε is not now restricted to values greater than unity; but if it should be greater than unity, we have again a hyperbola (Fig. 5·51) and the same relation,

$$\cos \theta_0 = -\frac{1}{\varepsilon},$$

for the directions of its asymptotes as in the case illustrated in Fig. 5·5. In the present case, it will be observed, the orbit goes round the focus O, the centre of force. When $\varepsilon = 1$, the path of the particle becomes a parabola, the attracting centre being in the focus.

The case where ε is less than unity is of special interest. The orbit is now elliptic, the centre of force being in one of the foci. If a and b are the semi major and minor axes respectively of the ellipse, we have

$$-\frac{\Omega^2}{mB} = \frac{b^2}{a} \quad . \quad . \quad . \quad . \quad (5\cdot532)$$

The force exerted on one another by two gravitating particles is

$$-\frac{GMm}{r^2} \quad . \quad . \quad . \quad . \quad (5\cdot533)$$

where M and m are the masses of the particles, r the distance between them and G is the constant of gravitation. If the ratio $\frac{m}{M}$ is negligibly small compared with unity, the centre of mass of the pair of particles practically coincides with the particle M
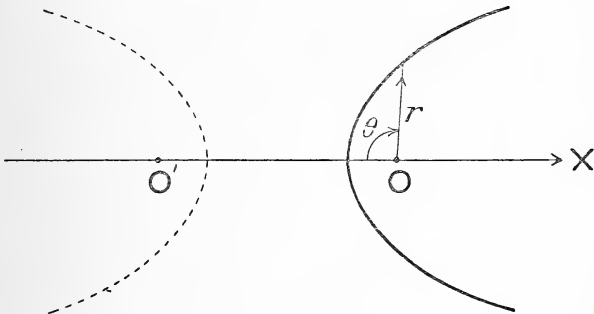
Fig. 5·51

and if we use axes of co-ordinates with the origin in the centre of mass we shall have the centre of attraction fixed in the origin.

The force exerted on m is $\frac{B}{r^2}$ where

$$B = -GMm \quad . \quad . \quad . \quad . \quad (5\cdot534)$$

by (5·533). It follows from (5·532) that

$$\frac{\Omega^2}{GMm^2} = \frac{b^2}{a} \quad . \quad . \quad . \quad . \quad (5\cdot535)$$

We have seen (§ 5·4) that Ω is equal to the product of m and twice the area swept out by the radius vector in the unit time. Therefore

$$\Omega = 2m\frac{\pi ab}{\tau} \quad . \quad . \quad . \quad . \quad (5\cdot536)$$

if τ is the period of a complete revolution. From (5·535) and (5·536) we get

$$\frac{a^3}{\tau^2} = \frac{GM}{4\pi^2} \quad . \quad . \quad . \quad . \quad . \quad (5·54)$$

Johannes Kepler (1571–1630) inferred from the observational data accumulated by Tycho Brahé (1546–1601) that the planets travel round the sun in elliptic orbits, the sun being in one focus of the ellipse, and that the radii vectores sweep out equal areas in equal times. These inferences, known as Kepler's first and second laws, were published in 1609.

Equation 5·54 expresses Kepler's third law, namely that the cubes of the major axes of different planetary orbits are proportional to the squares of the corresponding periods of revolution. This was published by him a few years later.

A rather important proposition connected with elliptic motion under the inverse square law of force is the following :—The energy of the particle m is completely determined by the length of the major axes of the ellipse.   The energy equation (5·51) reduces to

$$\frac{\Omega^2}{2mr^2} + \frac{B}{r} = E,$$

or
$$r^2 - \frac{B}{E}r - \frac{\Omega^2}{2mE} = 0 \quad . \quad . \quad . \quad . \quad (5·55)$$

when the planetary body, electron or whatever it may be, is just at one or other end of the major axis, since in this case $\frac{dr}{dt} = 0$. If the corresponding values of the radius vector are r_1 and r_2, we have of course

$$r_1 + r_2 = 2a ;$$

but from (5·55) we have

$$r_1 + r_2 = \frac{B}{E},$$

and therefore

$$E = \frac{B}{2a} \quad . \quad . \quad . \quad . \quad . \quad (5·551)$$

which proves the proposition.

We have so far supposed the attracting (or repelling) centre to be fixed in space, or at all events fixed relatively to the axes of co-ordinates. We have now to amend equation (5·51) so that it will apply more generally to two mutually attracting or repelling bodies, not subject to other forces. Let M and m be their masses and R and r their respective distances from their

common centre of mass in which we suppose the origin to be placed. The energy equation now becomes

$$\tfrac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \tfrac{1}{2}M\left(\frac{dR}{dt}\right)^2 + \tfrac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2 + \tfrac{1}{2}MR^2\left(\frac{d\theta}{dt}\right)^2$$

$$+ \frac{B}{r+R} = E \quad . \quad . \quad . \quad . \quad (5\cdot56)$$

Since r and R are the distances of the masses m and M respectively from the centre of mass,

$$rm = RM,$$

or

$$\frac{m}{M} = \frac{R}{r} = \sigma$$

and therefore (5·56) becomes

$$\tfrac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \tfrac{1}{2}\left(\frac{m}{\sigma}\right)\sigma^2\left(\frac{dr}{dt}\right)^2$$

$$+ \left(\tfrac{1}{2}mr^2 + \tfrac{1}{2}\frac{m}{\sigma}\sigma^2r^2\right)\left(\frac{d\theta}{dt}\right)^2 + \frac{B}{r(1+\sigma)} = E.$$

Therefore

$$\tfrac{1}{2}m(1+\sigma)\left(\frac{dr}{dt}\right)^2 + \tfrac{1}{2}mr^2(1+\sigma)\left(\frac{d\theta}{dt}\right)^2 + \frac{B}{r(1+\sigma)} = E \quad (5\cdot561)$$

This equation would be identical with (5·51) if in the latter m were replaced by $m' = m(1 + \sigma)$ and B by $B' = \dfrac{B}{1+\sigma}$; so that the problem of the motion of the particle m is reduced to the one we have already solved. In particular we find for the energy of the system

$$E = \frac{B'}{2a} = \frac{B}{2a} \cdot \frac{M}{M+m}$$

where a is the semi-major axis of the ellipse in which the particle m is moving.

An instructive example of a central force is that represented by $\dfrac{B}{r^2} + \dfrac{C}{r^3}$, where B and C are constants. The potential energy is obviously

$$\frac{B}{r} + \frac{C}{2r^2},$$

if we adopt, as usual, the value zero for the arbitrary constant involved. The energy equation is now

$$\tfrac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{\Omega^2}{2mr^2} + \frac{B}{r} + \frac{C}{2r^2} = E \quad . \quad . \quad . \quad (5\cdot57)$$

5

and differs from (5·51) only in the expression for the potential energy. The same method as that which was used in the case of the inverse square law leads to

$$\left(\frac{du}{d\theta}\right)^2 + \left(1 + \frac{mC}{\Omega^2}\right)u^2 + \frac{2mB}{\Omega^2}u = \frac{2mE}{\Omega^2} \quad . \quad (5\text{·}571)$$

instead of (5·514), and to

$$\frac{d^2u}{d\theta^2} + \left(1 + \frac{mC}{\Omega^2}\right)u = -\frac{mB}{\Omega^2} \quad . \quad . \quad . \quad (5\text{·}58)$$

instead of (5·52).

We shall restrict our attention to the case where $1 + \dfrac{mC}{\Omega^2}$ is positive and not zero. The general solution of (5·58) may then be expressed in the form

$$u = -\frac{mB}{(\Omega^2 + mC)} + \mathrm{R} \cos\left\{\sqrt{1 + \frac{mC}{\Omega^2}}\,\theta - \eta\right\} \quad (5\text{·}59)$$

where R and η are constants of integration. We may write this equation in the form

$$r = \frac{-\dfrac{\Omega^2 + mC}{mB}}{1 + \varepsilon \cos\left\{\sqrt{1 + \dfrac{mC}{\Omega^2}}\Big|\theta - \eta\right\}}, \quad . \quad . \quad (5\text{·}591)$$

where both ε and $\sqrt{1 + \dfrac{mC}{\Omega^2}}\Big|$ may, without loss of generality, be taken to be positive. It will be seen that (5·591) resembles the equation of a conic. This resemblance is made more obvious still if we write it in the following way:

$$r = \frac{-\dfrac{\Omega^2 + mC}{mB}}{1 + \varepsilon \cos\{\theta - \eta'\}}, \quad . \quad . \quad . \quad (5\text{·}592)$$

where

$$\eta' = \theta\left[1 - \sqrt{1 + \frac{mC}{\Omega^2}}\Big|\right] + \eta.$$

The essential difference between the orbit represented by (5·531) and that represented by (5·592) lies in the fact that whereas in the former η is a constant the corresponding quantity η' in the latter is continuously increasing or decreasing during the motion, according as $\sqrt{1 + \dfrac{mC}{\Omega^2}}\Big|$ is less than or greater than unity. If we give the axes of co-ordinates a suitably adjusted motion

of rotation about the Z axis we may keep η' *constant* and evidently the orbit *referred to such moving axes will be a conic section*. We may say therefore that the orbit is a conic section the major axis of which is in rotation, in the plane of the motion, in the same direction as that in which the particle is moving, or in the opposite one, according as $\sqrt{\left|1 + \dfrac{mC}{\Omega^2}\right|}$ is less than or greater than unity.

§ 6.   Generalized Co-ordinates

The n independent numerical data, $q_1, q_2, q_3, \ldots q_n$, which are necessary for the complete specification of the configuration of a dynamical system, are called its **generalized co-ordinates**, and the system is said to have n degrees of freedom.   Such co-ordinates may be chosen in various ways; for ex-ample in the case of a single particle they may be its rectangular or polar co-ordinates.   A rigid body, free to move in any way, will require six co-ordin-ates ; but only three if a single point in the body is fixed.   Associated with each q is a **generalized velocity** $\dfrac{dq}{dt}$, and a **gene-**
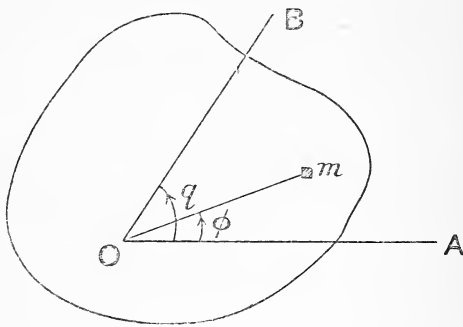
Fig. 6

ralized momentum usually represented by the letter p.   A rigid body, the only possible motion of which is a rotation about a fixed axis, has only one q, which may conveniently be the angle between a fixed plane of reference, OA in Fig. 6, and another plane, OB, fixed in the body, the intersection of the planes coinciding with O, the axis of rotation, which in the figure is perpendicular to the plane of the paper.   The kinetic energy of any particle of mass m in the body is

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mr^2\left(\frac{d\phi}{dt}\right)^2,$$

where r is the perpendicular distance of the particle from the axis, O, and ϕ is the angle between r and OA.   If the body is rigid

$$\frac{d\phi}{dt} = \frac{dq}{dt},$$

and the kinetic energy of the particle is

$$\tfrac{1}{2}mr^2\left(\frac{dq}{dt}\right)^2.$$

Since $\left(\dfrac{dq}{dt}\right)^2$ has the same value for all particles in the body, the total kinetic energy will be

$$T = \tfrac{1}{2}(\Sigma mr^2)\left(\frac{dq}{dt}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad \textbf{(6)}$$

The quantity Σmr^2 is called the **moment of inertia** of the body with respect to the axis in question, and it occupies in formula (6) the same position as that of the mass in the expression $\tfrac{1}{2}mv^2$ for the kinetic energy of a particle.

The generalized momentum corresponding to $\dfrac{dq}{dt}$ is defined by

$$p = \Sigma mr^2 . \frac{dq}{dt} \quad . \quad . \quad . \quad . \quad . \quad \textbf{(6·01)}$$

This definition is in accord with the use of the term *angular momentum* in § 5·4 (see equation **5·412**). If the axis of rotation of a rigid body pass through the origin and if x, y and z are the co-ordinates of any particle of mass m, in the body, the components of the angular momentum of the body will be

$$p_x = \Sigma m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right),$$

$$p_y = \Sigma m\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right),$$

$$p_z = \Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) \quad . \quad . \quad . \quad \textbf{(6·011)}$$

see equations **(5·41)**.

Equations **(5·432)** also will apply to a rigid body and may be expressed in the form

$$\frac{dp_x}{dt} = \Sigma(yF_z - zF_y),$$

$$\frac{dp_y}{dt} = \Sigma(zF_x - xF_z),$$

$$\frac{dp_z}{dt} = \Sigma(xF_y - yF_x) \quad . \quad . \quad . \quad . \quad \textbf{(6·02)}$$

In these equations we may, just as in § 5·43, regard the forces as of external origin, since the forces of internal origin contribute nothing to $\Sigma(yF_z - zF_y)$, etc. The right-hand members of (6·02) are the components of the applied **torque** or **couple.**

The parallelism between equations (**6·02**) and those for the motion of a particle should be noticed. The former state that **Rate of Increase of Angular Momentum = Applied Couple,** and the latter that

Rate of Increase of Momentum = Applied Force.

§ 6·1. WORK AND ENERGY

Let $\mathbf{dq} \equiv (dq_x,\ dq_y,\ dq_z)$ be a small rotation of the body about an axis through the origin. We may represent $\mathbf{dq}$ by a straight line from the origin, of length $\mathbf{dq}$ and drawn in the direction in which an ordinary screw would travel with such a rotation. If $\mathbf{dl} \equiv (dx,\ dy,\ dz)$ is the consequent displacement of a particle, the distance of which from the origin is $\mathbf{r} \equiv (x, y, z)$. We have, numerically,

$$\mathbf{dl} = \mathbf{r} \sin \theta\ \mathbf{dq},$$

where θ is the angle between the directions of $\mathbf{dq}$ and $\mathbf{r}$; and when we study the directions we find

$$\mathbf{dl} = [\mathbf{dq, r}] \quad . \quad . \quad . \quad . \quad . \quad (6 \cdot 1)$$

The work done during the displacement $\mathbf{dl}$ is equal to the scalar product of $\mathbf{F}$, the force acting on the particle, and $\mathbf{dl}$ the displacement. Therefore the work is

$$(\mathbf{dl}\ \mathbf{F}) = ([\mathbf{dq\ r}]\mathbf{F}).$$

Reference to equation (**2·14**) will show that this is equal to

$$([\mathbf{r\ F}]\ \mathbf{dq}).$$

In a rigid body $\mathbf{dq}$ is the same for all particles, therefore the work done by all the forces acting on the body will be given by

$$(\mathbf{dq}\ \Sigma\,[\mathbf{r\ F}]),$$

and the rate at which work is done will be equal to

$$\left(\frac{\mathbf{dq}}{dt},\ \Sigma[\mathbf{r\ F}]\right).$$

This must be equal to the rate of increase of the kinetic energy of the body, so that we have

$$\frac{dT}{dt} = \left(\frac{\mathbf{dq}}{dt},\ \Sigma[\mathbf{r\ F}]\right) \quad . \quad . \quad . \quad (6 \cdot 11)$$

In the summation we need only take account of forces of external origin since the contribution to $\Sigma[\mathbf{r\ F}]$ of the mutual forces exerted by the particles of the body on one another is zero. Once more there is a close parallelism with a corresponding result in particle dynamics. Equation (**6·11**) states that the rate of increase of the kinetic energy is equal to the scalar

product of angular velocity and applied couple. The rate of increase of the kinetic energy of a particle is equal to the scalar product of its velocity and the force acting on it.

§ 6·2. Moments and Products of Inertia

If I_0 and I_c represent respectively the moments of inertia of a rigid body with respect to any axis, O, and an axis C, through the centre of mass of the body and parallel to O,

$$I_0 = Mh^2 + I_c \quad . \quad (6\cdot2)$$

Fig. 6·2

M being the mass of the body and h the perpendicular distance between the two axes. To prove this it is convenient to place the co-ordinate axes so that the axes O and C are in the XZ plane and perpendicular to the X axis (Fig. 6·2). The contribution of any particle m to I_0 is

$$mr^2 = mh^2 + ms^2 - 2mhs \cos \phi$$

or

$$mr^2 = mh^2 + ms^2 + 2hm(x - x_0),$$

where x is the X co-ordinate of the particle and x_0 is the X co-ordinate of the centre of mass. Summing over all the particles in the body we get

$$I_0 = Mh^2 + I_c + 2h\{\Sigma mx - Mx_0\},$$

and, by the definition of centre of mass,

$$Mx_0 = \Sigma mx,$$

therefore equation (6·2) follows.

The **radius of gyration** of a body with respect to any axis is defined to be the positive square root of the quotient of the moment of inertia of the body with respect to that axis by the mass of the body :

$$k = \sqrt{\frac{I}{M}},$$

so that

$$I = Mk^2 \quad . \quad . \quad . \quad . \quad . \quad (6\cdot21)$$

Unless the contrary is expressly stated, or clearly implied by the context, we shall associate the radius of gyration with axes through the centre of mass of the body.

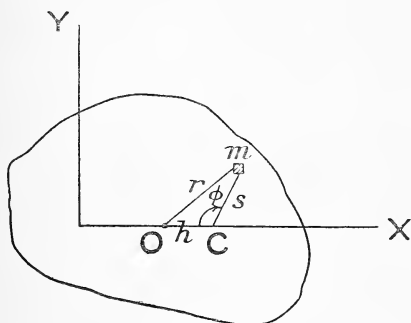Let α, β and γ be the direction cosines of an axis through

the centre of mass, which we shall suppose is at the origin of rectangular co-ordinates. The contribution of any particle of mass m to the moment of inertia, $I_{\alpha\beta\gamma}$, with respect to the axis (α, β, γ) is $m\mathbf{r}^2 \sin^2 \theta$, where $\mathbf{r} \equiv (x, y, z)$ specifies the position of the particle and θ is the angle between the directions of $\mathbf{r}$ and of (α, β, γ). Since $\mathbf{r}^2 \sin^2 \theta$ is the square of the vector product of the vector $\mathbf{r}$ and the unit vector (α, β, γ),

$$m\mathbf{r}^2 \sin^2 \theta = m\{(y\gamma - z\beta)^2 + (z\alpha - x\gamma)^2 + (x\beta - y\alpha)^2\}.$$

Therefore

$$I_{\alpha\beta\gamma} = \alpha^2 \Sigma m(y^2 + z^2) + \beta^2 \Sigma m(x^2 + z^2) + \gamma^2 \Sigma m(x^2 + y^2)$$
$$- 2\beta\gamma \Sigma myz - 2\alpha\gamma \Sigma mxz - 2\alpha\beta \Sigma mxy,$$

$$I_{\alpha\beta\gamma} = A\alpha^2 + B\beta^2 + C\gamma^2 + 2D\beta\gamma + 2E\alpha\gamma + 2F\alpha\beta \quad . \quad (6\cdot22)$$

The coefficients

$$A = \Sigma m(y^2 + z^2),$$
$$B = \Sigma m(x^2 + z^2)$$

and

$$C = \Sigma m(x^2 + y^2)$$

are, clearly, the moments of inertia of the body with respect to the X, Y and Z axes respectively. The remaining coefficients

$$- D = \Sigma myz,$$
$$- E = \Sigma mxz$$

and

$$- F = \Sigma mxy$$

are known as products of inertia.

Consider the surface

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2D\eta\zeta + 2E\xi\zeta + 2F\xi\eta = M \quad . \quad (6\cdot23)$$

where M is the mass of the body and ξ, η and ζ are X, Y and Z co-ordinates. Let ϱ be the length of a radius vector from the origin to a point $(\xi\eta\zeta)$ on the surface, so that

$$\varrho^2 = \xi^2 + \eta^2 + \zeta^2.$$

We have for a radius vector in the direction $(\alpha\beta\gamma)$

$$\alpha = \xi/\varrho, \ \beta = \eta/\varrho, \ \gamma = \zeta/\varrho,$$

and therefore, dividing both sides of $(6\cdot23)$ by ϱ^2,

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2D\beta\gamma + 2E\alpha\gamma + 2F\alpha\beta = \frac{M}{\varrho^2},$$

or

$$I_{\alpha\beta\gamma} = Mk^2 = \frac{M}{\varrho^2} ;$$

so that

$$k = \frac{1}{\varrho}.$$

This means that the length of any radius vector of the surface

(**6·23**) is equal to the reciprocal of the radius of gyration of the body with respect to an axis in that direction.

The equation (**6·23**) must be that of an ellipsoid, since in any direction whatever ϱ will have a finite positive value. It is called the **momental ellipsoid** or **ellipsoid of inertia**. A suitable rotation of the axes about the origin will reduce the equation to

$$A\xi^2 + B\eta^2 + C\zeta^2 = M \quad . \quad . \quad . \quad (6·235)$$

where A, B and C have not necessarily the same values as in (**6·23**). A, B and C (**6·235**) are called the **principal moments of inertia** of the body and the corresponding X, Y and Z directions are called the **principal axes of inertia**.

In equation (**6·235**) let us replace ξ, η and ζ by $\dfrac{M}{A}\xi'$, $\dfrac{M}{B}\eta'$ and $\dfrac{M}{C}\zeta'$ respectively, so that we get the equation

$$\frac{\xi'^2}{A} + \frac{\eta'^2}{B} + \frac{\zeta'^2}{C} = \frac{1}{M}$$

or

$$\frac{\xi'^2}{k_1^2} + \frac{\eta'^2}{k_2^2} + \frac{\zeta'^2}{k_3^2} = 1 \quad . \quad . \quad . \quad . \quad (6·24)$$

in which k_1, k_2 and k_3 are the radii of gyration about the principal axes of inertia. The surface (**6·24**) is called the **ellipsoid of gyration**. Any point (ξ', η', ζ') on it corresponds to a point (ξ, η, ζ) on the momental ellipsoid, where

$$\xi = \frac{M}{A}\xi',$$

$$\eta = \frac{M}{B}\eta' \quad . \quad . \quad . \quad . \quad . \quad (6·241)$$

$$\zeta = \frac{M}{C}\zeta'.$$

The length of the perpendicular, P, from the origin to the tangent plane at $(\xi'\ \eta'\ \zeta')$ is, by a well-known rule,

$$P = \frac{\dfrac{1}{M}}{\sqrt{\dfrac{\xi'^2}{A^2} + \dfrac{\eta'^2}{B^2} + \dfrac{\zeta'^2}{C^2}}},$$

or

$$P = \frac{1}{\sqrt{\xi^2 + \eta^2 + \zeta^2}} = \frac{1}{\varrho}, \text{ by} \quad . \quad . \quad (6·241)$$

and it is easily seen that its direction cosines are the same as those of the radius vector ϱ from the origin to the point $(\xi\ \eta\ \zeta)$

on the momental ellipsoid. It follows that the length of the perpendicular from the origin to a tangent plane of the ellipsoid of gyration is equal to the radius of gyration of the body about an axis coincident with the perpendicular.

§ 6·3. The Momental Tensor

Equation (6·22) may be written in the form :

$$\alpha\{A\alpha + F\beta + E\gamma\} + \beta\{F\alpha + B\beta + D\gamma\} \\ + \gamma\{E\alpha + D\beta + C\gamma\} = I_{\alpha\beta\gamma} \quad . \quad . \quad . \quad (6·3)$$

The right-hand member of (6·3) is an invariant, since it is the moment of inertia of the body with respect to a specified axis in it and must therefore be independent of the system of reference chosen. This suggests that the set of quantities

$$\begin{matrix} A & F & E \\ F & B & D \\ E & D & C \end{matrix} \quad . \quad . \quad . \quad . \quad (6·301)$$

are the components of a tensor of the second rank. It can easily be proved that this is the case (see § 2·3). It is called the **momental tensor** and is more appropriately represented in the following way :

$$\begin{matrix} i_{xx} & i_{xy} & i_{xz} \\ i_{yx} & i_{yy} & i_{yz} \\ i_{zx} & i_{zy} & i_{zz} \end{matrix} \quad . \quad . \quad . \quad . \quad (6·302)$$

If we make a corresponding change of notation for the direction cosines $(\alpha\beta\gamma)$, equation (6·3), becomes :

$$\begin{aligned} I_{\alpha} = \quad & a_x\{a_x\,i_{xx} + a_y\,i_{xy} + a_z\,i_{xz}\} \\ + \; & a_y\{a_x\,i_{yx} + a_y\,i_{yy} + a_z\,i_{yz}\} \\ + \; & a_z\{a_x\,i_{zx} + a_y\,i_{zy} + a_z\,i_{zz}\} \quad . \quad . \quad (6·303) \end{aligned}$$

The moment of inertia I_{α} is therefore to be regarded as the scalar product of two vectors, namely the vector $(a_x\,a_y\,a_z)$ and the vector, the components of which are represented by the expressions in brackets in (6·303).

§ 6·4. Kinetic Energy of a Rigid Body

If the motion of the body is a rotation about an axis α, the angular velocity being $\boldsymbol{\omega} = \dfrac{d\mathbf{q}}{dt}$, its kinetic energy is $T = \frac{1}{2}I_{\alpha}\omega^2$ and since $\omega_x = \omega a_x$, $\omega_y = \omega a_y$, $\omega_z = \omega a_z$, we obtain from (6·303)

$$\begin{aligned} T = \quad & \tfrac{1}{2}\omega_x\{\omega_x\,i_{xx} + \omega_y\,i_{xy} + \omega_z\,i_{xz}\} \\ + \; & \tfrac{1}{2}\omega_y\{\omega_x\,i_{yx} + \omega_y\,i_{yy} + \omega_z\,i_{yz}\} \\ + \; & \tfrac{1}{2}\omega_z\{\omega_x\,i_{zx} + \omega_y\,i_{zy} + \omega_z\,i_{zz}\} \quad . \quad . \quad (6·4) \end{aligned}$$

If the axes of co-ordinates coincide with the principal axes of inertia, this becomes

$$T = \tfrac{1}{2}\{\omega_x{}^2\, i_{xx} + \omega_y{}^2\, i_{yy} + \omega_z{}^2\, i_{zz}\};$$

or, if we revert to the notation of (6·235) for the principal moments of inertia,

$$T = \tfrac{1}{2}\{A\omega_x{}^2 + B\omega_y{}^2 + C\omega_z{}^2\} \quad . \quad . \quad . \quad (6·41)$$

If in (6·4) we represent the angular velocity $\boldsymbol{\omega}$ by $\dfrac{d\mathbf{q}}{dt}$, we have

$$2T = p_x\frac{dq_x}{dt} + p_y\frac{dq_y}{dt} + p_z\frac{dq_z}{dt};$$

where

$$p_x = i_{xx}\frac{dq_x}{dt} + i_{xy}\frac{dq_y}{dt} + i_{xz}\frac{dq_z}{dt},$$

$$p_y = i_{yx}\frac{dq_x}{dt} + i_{yy}\frac{dq_y}{dt} + i_{yz}\frac{dq_z}{dt},$$

$$p_z = i_{zx}\frac{dq_x}{dt} + i_{zy}\frac{dq_y}{dt} + i_{zz}\frac{dq_z}{dt},$$

are the components of the angular momentum. In fact the X component of the angular momentum is

$$p_x = \Sigma m\left\{y\frac{dz}{dt} - z\frac{dy}{dt}\right\},$$

and

$$\frac{dy}{dt} = \omega_z x - \omega_x z,$$

$$\frac{dz}{dt} = \omega_x y - \omega_y x,$$

(see equation (6·1) for example); therefore

$$p_x = \Sigma m\{\omega_x y^2 - \omega_y xy - \omega_z xz + \omega_x z^2\}$$
$$= i_{xx}\omega_x + i_{xy}\omega_y + i_{xz}\omega_z.$$

It will be observed that

$$p_x = \frac{\partial T}{\partial \dot{q}_x},\ p_y = \frac{\partial T}{\partial \dot{q}_y},\ p_z = \frac{\partial T}{\partial \dot{q}_z};$$

where

$$\dot{q}_x = \omega_x = \frac{dq_x}{dt},\ \text{etc.} \quad . \quad . \quad . \quad . \quad (6·42)$$

Any motion of a rigid body can be regarded as a motion of translation, in which all the particles of the body receive equal and parallel displacements, on which is superposed a rotation about a suitably chosen axis. Let $\mathbf{r}$ be the distance of any particle of mass m from a point, P, on the axis of rotation. A rotation $\mathbf{dq}$ will give it a displacement $[\mathbf{dq}\ \mathbf{r}]$. The total displacement of the particle will be the vector sum of $[\mathbf{dq}\ \mathbf{r}]$

and the displacement of P. Let the co-ordinates of the particle be x, y and z and those of P be x_0, y_0 and z_0; then

$$\mathbf{r} \equiv (x - x_0,\ y - y_0,\ z - z_0),$$

and

$$dx = dx_0 + \{dq_y(z - z_0) - dq_z(y - y_0)\} \qquad . \quad (6\cdot43)$$

Therefore

$$v_x = v_{0x} + v_{1x} \qquad . \quad . \quad . \quad . \quad (6\cdot435)$$

Where $\mathbf{v}$ is the velocity of the particle, $\mathbf{v}_0$ the velocity of P and

$$v_{1x} = \frac{dq_y}{dt}(z - z_0) - \frac{dq_z}{dt}(y - y_0) \quad . \quad . \quad (6\cdot436)$$

The kinetic energy of the particle is

$$\tfrac{1}{2}m\mathbf{v}^2 = \tfrac{1}{2}m(v_x{}^2 + v_y{}^2 + v_z{}^2)$$
$$= \tfrac{1}{2}m\mathbf{v}_0{}^2 + \tfrac{1}{2}m\mathbf{v}_1{}^2 + m(v_{0x}v_{1x} + v_{0y}v_{1y} + v_{0z}v_{1z}),$$

and the kinetic energy of the body,

$$T = T_0 + T_1 + \Sigma m(v_{0x}v_{1x} + v_{0y}v_{1y} + v_{0z}v_{1z}) \quad . \quad (6\cdot44)$$

In this equation

$$T_0 = \tfrac{1}{2}M v_0{}^2,$$
$$T_1 = \tfrac{1}{2}I\omega^2,$$

M is the mass of the body and I is its moment of inertia with respect to the axis of rotation.

If the point P is the centre of mass of the body,

$$\Sigma m(v_{0x}v_{1x} + v_{0y}v_{1y} + v_{0z}v_{1z}) = 0,$$

since, as reference to (6·436) will show, it consists of terms, each of which contains one or other of the factors

$$\Sigma m(z - z_0),\ \Sigma m(y - y_0),\ \text{etc.},$$

all of which vanish if $(x_0\ y_0\ z_0)$ is the centre of mass. We thus arrive at the important result,

$$T = \tfrac{1}{2}M\mathbf{v}_0{}^2 + \tfrac{1}{2}I\boldsymbol{\omega}^2 \quad . \quad . \quad . \quad . \quad (6\cdot45)$$

where $\mathbf{v}_0$ is the velocity of the centre of mass and $\boldsymbol{\omega}$ is the angular velocity relative to the centre of mass.

§ 6·5. THE PENDULUM

The pendulum is usually a rigid body mounted so that it can turn freely about a fixed horizontal axis, O, (Fig. 6·5), which we may suppose to be the Z axis of rectangular co-ordinates. The position of the pendulum is determined by the vector

$$\mathbf{q} \equiv (q_x\ q_y\ q_z),$$

where $q_x = q_y = 0$ and $q_z = q$ is the angle between the plane XZ and the plane, OC, containing the centre of mass and the

axis of rotation. The positive direction of q is indicated in the figure by an arrow. The equations (**6·02**), when applied to this case reduce to

$$I\frac{d^2q}{dt^2} = \Sigma(xF_y - yF_x)$$

or

$$I\frac{d^2q}{dt^2} = \Sigma xF_y,$$

if the Y axis is directed vertically downwards, since the impressed forces on the body are due to gravity only. The force F_y on any particle is equal to mg, therefore

$$I\frac{d^2q}{dt^2} = g\Sigma mx = Mgx_0,$$

or

$$-I\frac{d^2\theta}{dt^2} = Mgh \sin \theta,$$

where θ is the angle between OC and the vertical and h is the distance of the centre of mass from the axis of rotation. It is convenient to write the equation in the form:
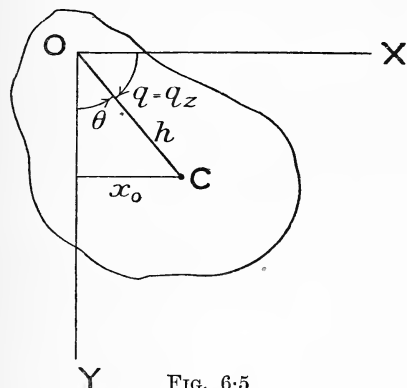
FIG. 6·5

$$\frac{d^2\theta}{dt^2} + \frac{Mgh}{I} \sin \theta = 0 \quad . \quad . \quad . \quad . \quad (6·5)$$

If we multiply by $\frac{d\theta}{dt}$ and integrate, we get

$$\tfrac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{Mgh}{I} \cos \theta = K \quad . \quad . \quad (6·501)$$

where K is a constant of integration. If K exceeds $\frac{Mgh}{I}$, the kinetic energy, $\tfrac{1}{2}I\left(\frac{d\theta}{dt}\right)^2$ can never sink to zero, and the body will keep on rotating in the same sense round the axis with a periodically varying angular velocity. The case of interest to us is that in which K is less than $\frac{Mgh}{I}$. There will then be a value θ_0 of θ between 0 and π for which $\frac{d\theta}{dt}$ is zero and

$$-\frac{Mgh}{I} \cos \theta_0 = K,$$

and consequently

$$\tfrac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \frac{Mgh}{I}\,(\cos\theta - \cos\theta_0),$$

or

$$\left(\frac{d\varepsilon}{dt}\right)^2 = \frac{Mgh}{I}\,(\sin^2\varepsilon_0 - \sin^2\varepsilon),$$

where

$$\varepsilon = \frac{\theta}{2}, \quad \varepsilon_0 = \frac{\theta_0}{2}.$$

The time required by the pendulum to travel from the position $\theta = 0$ to an extreme position $\theta = \pm\,\theta_0$ or from $\theta = \pm\,\theta_0$ to $\theta = 0$ is

$$\int dt = \sqrt{\frac{I}{Mgh}}\,\left|\,\int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\sin^2\varepsilon_0 - \sin^2\varepsilon}}\,\right|,$$

and therefore the complete period of oscillation is

$$\tau = 4\,\sqrt{\frac{I}{Mgh}}\,\left|\,\int_0^{\varepsilon_0} \frac{d\varepsilon}{\sqrt{\sin^2\varepsilon_0 - \sin^2\varepsilon}}\,\right| \qquad . \quad . \quad (6\cdot51)$$

To evaluate the integral we introduce a new variable, ϕ, defined by
$$\sin\varepsilon = \kappa\sin\phi,$$
where $\kappa = \sin\varepsilon_0$. On substituting in (6·51) we get

$$\tau = 4\,\sqrt{\frac{I}{Mgh}}\,\left|\,\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \kappa^2\sin^2\phi}}\,\right| \qquad . \quad . \quad (6\cdot511)$$

The elliptic integral in (6·511) is now expanded by means of the binomial theorem, thus,

$$\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \kappa^2\sin^2\phi}} = \int_0^{\pi/2} d\phi\Big\{1 + \tfrac{1}{2}\kappa^2\sin^2\phi + \frac{1.3}{2.4}\kappa^4\sin^4\phi$$
$$+ \frac{1.3.5}{2.4.6}\kappa^6\sin^6\phi + \,.\,.\,.\,\Big\}.$$

This can be integrated term by term, by using the well-known reduction formula

$$\int_0^{\pi/2} \sin^{2n}\phi\,d\phi = \frac{2n-1}{2n}\int_0^{\pi/2} \sin^{2n-2}\phi\,d\phi.$$

We get, finally,

$$\tau = 2\pi\,\sqrt{\frac{I}{Mgh}}\,\left|\,\Big\{1 + (\tfrac{1}{2}\kappa)^2 + \Big(\frac{1.3}{2.4}\kappa^2\Big)^2 + \Big(\frac{1.3.5}{2.4.6}\kappa^3\Big)^2 \,.\,.\,.\,\Big\}\right.\quad (6\cdot52)$$

When the amplitude, θ_0, is small, i.e. when κ is small, the period,

$$\tau_0 = 2\pi \sqrt{\frac{I}{Mgh}} \Bigg| \qquad \text{.} \quad \text{.} \quad \text{.} \quad \text{.} \quad (6\cdot521)$$

is independent of the amplitude. This result might have been reached much more shortly by replacing $\sin \theta$ in (6·5) by θ when θ is small. The equation then becomes

$$\frac{d^2\theta}{dt^2} + \frac{Mgh}{I}\theta = 0,$$

or, if we write ω^2 for the positive quantity $\dfrac{Mgh}{I}$,

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \quad \text{.} \quad \text{.} \quad \text{.} \quad \text{.} \quad \text{.} \quad (6\cdot522)$$

The general solution of this equation can be put in the form

$$\theta = A \cos(\omega t - \phi) \quad \text{.} \quad \text{.} \quad \text{.} \quad \text{.} \quad (6\cdot523)$$

where A and ϕ are arbitrary constants. Since θ will repeat its values every time $\omega t - \phi$ increases by 2π; we must have

$$\{\omega(t + \tau) - \phi\} - (\omega t - \phi) = 2\pi$$
$$\omega\tau = 2\pi,$$

or

$$\tau = \frac{2\pi}{\omega} \quad \text{.} \quad \text{.} \quad \text{.} \quad \text{.} \quad \text{.} \quad (6\cdot524)$$

This is identical with (6·521) when ω is replaced by $\sqrt{\dfrac{Mgh}{I}}\Bigg|$.

The type of motion defined by (6·522) is called **simple harmonic motion**. It has the important property that the period is independent of the amplitude.

By making use of (6·2) we may give to (6·521) the form

$$\tau_0 = 2\pi\sqrt{\frac{h^2 + k^2}{gh}}\Bigg| \qquad \text{.} \quad \text{.} \quad \text{.} \quad \text{.} \quad (6\cdot525)$$

In the ideal simple pendulum, $k = 0$, h is the distance, usually represented by l, from the point of support to the bob, and therefore, for small oscillations

$$\tau_0 = 2\pi\sqrt{\frac{l}{g}}\Bigg|.$$

If τ_1 represents the still better approximation obtained by ignoring quantities of the order of κ^4 and higher powers of κ, we have from (6·52)

$$\tau_1 = \tau_0\left(1 + \frac{\kappa^2}{4}\right),$$

or since $\kappa = \sin \dfrac{\theta_0}{2}$,

$$\tau_1 = \tau_0\left(1 + \tfrac{1}{4}\sin^2\frac{\theta_0}{2}\right),$$

or
$$\tau_1 = \tau_0\left(1 + \frac{\theta_0{}^2}{16}\right) \quad . \quad . \quad . \quad . \quad . \quad (6\cdot526)$$

since the difference between the squares of $\sin\dfrac{\theta_0}{2}$ and $\dfrac{\theta_0}{2}$ is of the order of κ^4.

Cycloidal Pendulum. We have seen that the period of the type of pendulum we have been studying is a function of the amplitude. It was shown by Huygens (*Horologium Oscillatorium*) that the motion of a particle, constrained to travel along a certain cycloid, is strictly isochronous, i.e. the period is independent of the amplitude. The equations of motion of a particle, P, constrained to travel along a curve in a vertical plane, the XY plane in Fig. 6·51, are

$$m\frac{d^2x}{dt^2} = Q_x,$$

$$m\frac{d^2y}{dt^2} = mg + Q_y \quad (6\cdot53)$$
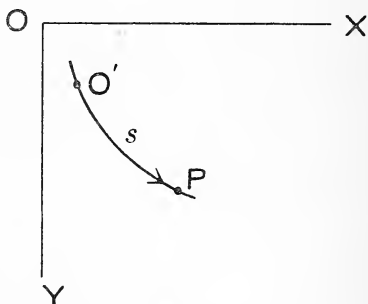
Fig. 6·51

where Q_x and Q_y are the components of the constraining force. If s is the distance travelled by the particle along the curve, measured from some arbitrarily chosen point, O', the vector $\mathbf{ds} \equiv (dx,\, dy)$ is perpendicular to the vector $\mathbf{Q}$. Therefore

$$Q_x dx + Q_y dy = 0 \quad . \quad . \quad . \quad . \quad (6\cdot531)$$

Multiply the equations (6·53) by $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ respectively and add,

$$m\frac{dx}{dt}\frac{d^2x}{dt^2} + m\frac{dy}{dt}\frac{d^2y}{dt^2} = mg\frac{dy}{dt} + Q_x\frac{dx}{dt} + Q_y\frac{dy}{dt},$$

or
$$\frac{m}{2}\frac{d}{dt}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} = mg\frac{dy}{dt},$$

by equation (6·531), and therefore

$$\frac{m}{2}\frac{d}{dt}\left\{\left(\frac{ds}{dt}\right)^2\right\} = mg\cos\varepsilon\frac{ds}{dt},$$

if ε is the angle between dy and ds. Dividing through by $m\dfrac{ds}{dt}$ we have

$$\frac{d^2s}{dt^2} = g \cos \varepsilon \quad . \quad . \quad . \quad . \quad . \quad (6\cdot54)$$

The motion of the particle will be simple harmonic (see the definition **6·522**), and its period consequently independent of the amplitude, if

$$\cos \varepsilon = -\, as . \quad . \quad . \quad . \quad . \quad (6\cdot541)$$

where a is any positive constant. Equation (**6·54**) then becomes

$$\frac{d^2s}{dt^2} + ags = 0,$$

and the period of the motion is seen to be

$$\tau = \frac{2\pi}{\sqrt{ag}\,|}.$$

If l be the length of the simple pendulum, the small oscillations of which have the same period,

$$a = \frac{1}{l}$$

and (**6·541**) may be written as

$$\cos \varepsilon = -\, \frac{s}{l} \quad . \quad . \quad . \quad . \quad . \quad (6\cdot542)$$

This is the equation of the required curve. On differentiating it we get

$$ds = l \sin \varepsilon \, d\varepsilon$$

and therefore

$$dx = l \sin^2 \varepsilon \, d\varepsilon,$$
$$dy = l \sin \varepsilon \cos \varepsilon \, d\varepsilon.$$

Consequently

$$dx = \frac{l}{4}(1 - \cos \theta)d\theta,$$

$$dy = \frac{l}{4} \sin \theta \, d\theta \, ;$$

where $\theta = 2\varepsilon$. On integrating we have

$$x = \frac{l}{4}(\theta - \sin \theta) + A,$$

$$y = \frac{l}{4} \cos \theta + B \quad . \quad . \quad . \quad . \quad (6\cdot543)$$

Let O', from which s is measured, coincide with the origin. O, so that $x = 0$ when $y = 0$; and suppose that the particle,

P, is moving vertically downwards when in this position, i.e.
$\varepsilon = \theta = 0$ when $x = 0$ and $y = 0$. For this position of the
particle, therefore, equations (**6·543**) become

$$0 = 0 + A,$$

$$0 = -\frac{l}{4} + B\ ;$$

and on substituting in (**6·543**) we have the familiar equations
of the cycloid

$$x = R(\theta - \sin \theta),$$
$$y = R(1 - \cos \theta) \quad . \quad . \quad . \quad . \quad . \quad (\mathbf{6·55})$$

in which R has been written for $\dfrac{l}{4}$.

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CHAPTER IV

DYNAMICS OF A RIGID BODY FIXED AT ONE POINT

§ 7. EULER'S DYNAMICAL EQUATIONS

LET **P** be any vector and P_x, P_y and P_z its components referred to rectangular axes of co-ordinates. Let P_x', P_y' and P_z' be the components of the same vector referred to a second set of rectangular co-ordinates, the origin, O', of which coincides with O, the origin of the first system. Therefore, by (**2·22**)

$$P_x = a_{11}P_x' + a_{21}P_y' + a_{31}P_z'$$

and $$\frac{dP_x}{dt} = \frac{d}{dt}\{a_{11}P_x' + a_{21}P_y' + a_{31}P_z'\} \quad . \quad . \quad (7)$$

We shall suppose the first set of co-ordinates to be fixed and the second set to be in motion about their common origin. The cosines a_{11}, a_{21}, etc., are then variable and (**7**) becomes

$$\frac{dP_x}{dt} = a_{11}\frac{dP_x'}{dt} + a_{21}\frac{dP_y'}{dt} + a_{31}\frac{dP_z'}{dt}$$

$$+ P_x'\frac{da_{11}}{dt} + P_y'\frac{da_{21}}{dt} + P_z'\frac{da_{31}}{dt} \quad ((7\cdot001)$$

Now a_{11}, a_{21} and a_{31} are the co-ordinates in X', Y', Z' of a point on the X axis at the unit distance from the origin, and therefore $\frac{da_{11}}{dt}$, $\frac{da_{21}}{dt}$ and $\frac{da_{31}}{dt}$ are the components of the velocity of this point relatively to the *moving* axes. Therefore if ω_x', ω_y' and ω_z' are the components of the angular velocity of the fixed co-ordinates relatively to the moving co-ordinates,

$$\frac{da_{11}}{dt} = \omega_y'a_{31} - \omega_z'a_{21} = \omega_z a_{21} - \omega_y a_{31},$$

$$\frac{da_{21}}{dt} = \omega_z'a_{11} - \omega_x'a_{31} = \omega_x a_{31} - \omega_z a_{11},$$

$$\frac{da_{31}}{dt} = \omega_x'a_{21} - \omega_y'a_{11} = \omega_y a_{11} - \omega_x a_{21},$$

where $\omega_x = -\omega_x'$, $\omega_y = -\omega_y'$ and $\omega_z = -\omega_z'$, so that ω_x, ω_y and ω_z are the components of the angular velocity of the moving axes X', X', Z' relatively to the fixed axes X, X, Z.

If we now substitute these expressions for $\dfrac{da_{11}}{dt}$, etc., in equations (**7·001**) we get,

$$\frac{dP_x}{dt} = a_{11}\frac{dP_x'}{dt} + a_{21}\frac{dP_y'}{dt} + a_{31}\frac{dP_z'}{dt}$$
$$+ P_x'(\omega_z a_{21} - \omega_y a_{31}) + P_y'(\omega_x a_{31} - \omega_z a_{11})$$
$$+ P_z'(\omega_y a_{11} - \omega_x a_{21}) \quad . \quad . \quad (7\cdot002)$$

At an instant when the fixed and moving axes are coincident,
$$a_{11} = 1, \ a_{21} = a_{31} = 0$$
and equation (**7·002**) becomes

$$\frac{dP_x}{dt} = \frac{dP_x'}{dt} + P_z'\omega_y - P_y'\omega_z$$

to which we may add

$$\frac{dP_y}{dt} = \frac{dP_y'}{dt} + P'_x\omega_z - P_z'\omega_x \quad . \quad . \quad . \quad (7\cdot01)$$

$$\frac{dP_z}{dt} = \frac{dP_z'}{dt} + P_y'\omega_x - P_x'\omega_y.$$

It is very easy to be misled by these equations, and we shall therefore inquire carefully about their significance before applying them. In arriving at the transformation (**2·22**) we represented the vector concerned (in the present case **P**) by a straight line drawn from the origin in the direction of the vector, and having a length numerically equal to it. Therefore P_x, P_y and P_z are the co-ordinates of the end point of the line. Equations (**7·01**) apply at the instant when the two co-ordinate systems coincide. Hence $P_x = P_x'$, $P_y = P_y'$, $P_z = P_z'$. Suppose now that **P** is the angular velocity of a rigid body with one point fixed in the common origin of the co-ordinate systems. Clearly the components of the angular velocity of the body have the same values in both systems of co-ordinates when they happen to coincide. It is important to note this and so avoid the error of confusing the angular velocity *referred to the moving axes* with the angular velocity *relative to the moving axes*. In fact, if the moving axes were fixed in the rigid body, its angular velocity would be $(\omega_x, \omega_y, \omega_z)$ in both systems of co-ordinates; but obviously zero *relative* to the moving axes; and we note too that the rate of change of ω is the same whether referred to the fixed or the moving axes, as is immediately evident on substituting **ω** for **P** in (**7·01**).

Let us now suppose the moving axes to be fixed in a rigid body and to coincide with its principal axes of inertia through the fixed point of the body (the common origin of both systems of co-ordinates) and let us further suppose $\mathbf{P} \equiv (P_x, P_y, P_z)$ to be the angular momentum of the body. Then $\left(\dfrac{dP_x}{dt}, \dfrac{dP_y}{dt}, \dfrac{dP_z}{dt} \right)$ becomes the torque or couple applied to the body. In what follows we shall denote this by (L, M, N). At an instant when the axes are coincident

$$P_x = P_x' = A\omega_x,$$
$$P_y = P_y' = B\omega_y,$$
$$P_z = P_z' = C\omega_z,$$

where A, B and C are the principal moments of inertia of the body. Clearly

$$\frac{dP_x'}{dt} = A\frac{d\omega_x}{dt},$$

$$\frac{dP_y'}{dt} = B\frac{d\omega_y}{dt},$$

$$\frac{dP_z'}{dt} = C\frac{d\omega_z}{dt}.$$

On making these substitutions in equations (**7·01**) we obtain

$$L = A\frac{d\omega_x}{dt} + (C - B)\omega_y\omega_z,$$

$$M = B\frac{d\omega_y}{dt} + (A - C)\omega_z\omega_x,$$

$$N = C\frac{d\omega_z}{dt} + (B - A)\omega_x\omega_y \qquad . \quad . \quad . \quad (\mathbf{7·02})$$

When the applied couple vanishes these equations become

$$A\frac{d\omega_x}{dt} = (B - C)\omega_y\omega_z,$$

$$B\frac{d\omega_y}{dt} = (C - A)\omega_z\omega_x,$$

$$C\frac{d\omega_z}{dt} = (A - B)\omega_x\omega_y \quad . \quad . \quad . \quad (\mathbf{7·021})$$

The equations (**7·02**) and (**7·021**) are the well known **dynamical equations of Euler.**

On multiplying (**7·02**) by ω_x, ω_y, and ω_z respectively and adding, we get

$$\frac{d}{dt}\{\tfrac{1}{2}A\omega_x^2 + \tfrac{1}{2}B\omega_y^2 + \tfrac{1}{2}C\omega_z^2) = L\omega_x + M\omega_y + N\omega_z \qquad (\mathbf{7·03})$$

which states that the rate of increase of the kinetic energy of the body is equal to the rate at which the applied couple does work, a result we expect on other grounds (equation **6·11**).

When the couple applied to the body is zero, i.e. when $L = M = N = 0$, we find, by multiplying equations (**7·021**) by $A\omega_x$, $B\omega_y$ and $C\omega_z$ respectively and adding

$$\frac{d}{dt}\{\tfrac{1}{2}A^2\omega_x{}^2 + \tfrac{1}{2}B^2\omega_y{}^2 + \tfrac{1}{2}C^2\omega_z{}^2\} = 0,$$

or $\qquad\qquad A^2\omega_x{}^2 + B^2\omega_y{}^2 + C^2\omega_z{}^2 = \Omega^2 \quad . \quad . \quad .$ (**7·04**)

where Ω^2 is a constant. This equation is also to be anticipated on other grounds, since it expresses the constancy of the angular momentum (§ 6).

A particular solution of equations (**7·021**) is

$$\omega_x = \omega_y = 0 \; ; \quad \omega_z = \omega_0, \text{ a constant.}$$

This represents a rotation with constant angular velocity about a principal axis of inertia. Suppose the body to be rotating in this way and then slightly disturbed, so that it acquires very small angular velocities ω_x and ω_y about the other principal axes of inertia. How will it behave if it is now left to itself ? Since ω_x and ω_y are small (i.e. by comparison with ω_0), we shall ignore the product $\omega_x\omega_y$. Euler's equations now become

$$A\frac{d\omega_x}{dt} + (C - B)\omega_0\omega_y = 0 \quad . \quad . \quad . \quad . \quad (7\cdot05)$$

$$B\frac{d\omega_y}{dt} + (A - C)\omega_0\omega_x = 0.$$

Differentiating the former of these with respect to the time, and eliminating $\dfrac{d\omega_y}{dt}$, we obtain

$$\frac{d^2\omega_x}{dt^2} + \frac{(C - B)(C - A)}{AB}\omega_0{}^2 . \omega_x = 0 \quad . \quad (7\cdot051)$$

By differentiating the second of the equations (**7·05**) in a similar way we obtain

$$\frac{d^2\omega_y}{dt^2} + \frac{(B - C)(A - C)}{AB}\omega_0{}^2\omega_y = 0 \quad . \quad (7\cdot052)$$

The constant

$$\frac{(C - B)(C - A)}{AB}\omega_0{}^2$$

in both of these equations is *positive if the moment of inertia C is either greater than A and B or smaller than A and B.* In such a case

$$\omega_x = R \cos{(at - \phi)}$$
$$\omega_y = S \cos{(at - \psi)} \quad . \quad . \quad (7\cdot053)$$

where R and S are small *real* constants, ϕ and ψ are constants, and

$$a = \sqrt{\frac{(C - A)(C - B)}{AB}}\, \omega_0 \quad . \quad . \quad (7\cdot054)$$

We see therefore that the motion of rotation is stable since ω_x and ω_y never exceed in absolute value the small constants R and S.

It should be noticed that R, S, ϕ and ψ are not all independent. The reason for this is that equations $(7\cdot051)$ and $(7\cdot052)$, in the solutions of which they occur, are more general than the equations $(7\cdot05)$ with which we are really concerned, since they are obtained from the latter by differentiation. If we abbreviate by writing

$$p = at - \phi,$$
$$q = at - \psi,$$

and substitute the solutions $(7\cdot053)$ in equations $(7\cdot05)$, we get

$$\frac{\sin p}{\cos q} = \frac{(C - B)S\omega_0}{aAR},$$

$$\frac{\sin q}{\cos p} = \frac{(A - C)R\omega_0}{aBS} \quad . \quad . \quad . \quad . \quad (7\cdot055)$$

and therefore

$$\frac{\sin p \sin q}{\cos p \cos q} = -1,$$

in consequence of $(7\cdot054)$.

It follows that p and q differ by an odd multiple of $\dfrac{\pi}{2}$ and the solutions $(7\cdot053)$ may consequently be put in the form

$$\omega_x = R \cos (at - \phi),$$
$$\omega_y = S \sin (at - \phi) \quad . \quad . \quad . \quad (7\cdot056)$$

the first of the equations $(7\cdot055)$ now becomes

$$1 = \frac{(C - B)S\omega_0}{ARa},$$

whence we get

$$S = R \sqrt{\frac{A(C - A)}{B(C - B)}} \quad . \quad . \quad . \quad (7\cdot057)$$

If we represent the angular velocity $\boldsymbol{\omega}$ by a straight line drawn from the origin, equal in length to $\boldsymbol{\omega}$, and in such a direction that the co-ordinates of its end points are ω_x, ω_y and ω_z $(= \omega_0)$ respectively, we see that it describes a small cone in the body. The end point travels along the small ellipse with semi-axes R and S.

The form of equations (**7·021**) suggests that their general solution can be expressed in terms of elliptic functions. Consider the integral

$$\xi = \int_0^\theta \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}|}, \quad k^2 < 1,$$

which belongs to the class of integrals called elliptic integrals. The upper limit, θ, is termed the **amplitude** of ξ and may be denoted by **am** ξ. Therefore

$$\sin \theta = \sin\, am\, \xi,$$

or, in the usual notation

$$\sin \theta = sn\, \xi.$$

Similarly $\qquad \cos \theta = \cos am\, \xi = cn\, \xi.$

The function $\sqrt{1 - k^2 \sin^2 \theta}|$ is usually called $\triangle \theta$,

$$\triangle\theta = \triangle\, am\, \xi = dn\, \xi.$$

The three functions, $sn\xi$, $cn\xi$ and $dn\xi$ are called **elliptic functions**. The differential quotient of $sn\xi$ with respect to ξ is

$$\frac{d\, sn\, \xi}{d\, \xi} = \frac{d \sin \theta}{d\theta} \cdot \frac{d\theta}{d\xi},$$

$$\frac{d\, sn\, \xi}{d\, \xi} = \cos \theta . \sqrt{1 - k^2 \sin^2 \theta}|,$$

or $\qquad \dfrac{d\, sn\, \xi}{d\, \xi} = cn\, \xi\, dn\, \xi.$

Similarly

$$\frac{d\, cn\, \xi}{d\, \xi} = -\, sn\, \xi\, dn\, \xi,$$

$$\frac{d\, dn\, \xi}{d\, \xi} = -\, k^2\, sn\, \xi\, cn\, \xi . \quad . \quad . \quad . \quad (7\text{·}06)$$

These equations suggest, as a solution of (**7·021**),

$$\omega_x = \omega_1 sn\, (at - \phi),$$
$$\omega_y = \omega_2 cn\, (at - \phi),$$
$$\omega_z = \omega_0 dn\, (at - \phi) \quad . \quad . \quad . \quad . \quad (7\text{·}07)$$

where ω_1, ω_2, ω_0, a and ϕ are constants, which, as we shall see, are not all independent. Substituting in (**7·021**), we find

$$a\omega_1 = \frac{B - C}{A}\omega_2\omega_0$$

$$-\, a\omega_2 = \frac{C - A}{B}\omega_1\omega_0$$

$$-\, ak^2\omega_0 = \frac{A - B}{C}\omega_1\omega_2 \quad . \quad . \quad . \quad (7\text{·}071)$$

Therefore

$$\frac{\omega_1{}^2}{\omega_2{}^2} = \frac{(C-B)B}{(C-A)A},$$

$$k^2 = \frac{(A-B)A}{(C-B)C}\frac{\omega_1{}^2}{\omega_0{}^2} \quad . \quad . \quad . \quad (7\cdot072)$$

and

$$a^2 = \frac{(C-B)(C-A)}{AB}\omega_0{}^2.$$

Of the six constants, ω_1, ω_2, ω_0, k, a and ϕ therefore, three can be expressed in terms of the remaining three. These latter may be chosen arbitrarily and the solution (**7·07**) is therefore the general one. Let us select ω_1, ω_0 and ϕ as the arbitrary constants and consider the case where ω_1 and ω_2 are very small compared with ω_0. The parameter k^2 will be a small quantity of the second order, by the second equation (**7·072**). We shall therefore ignore it. We thus get

$$\xi = \theta$$

in the equations defining the elliptic functions. Therefore

$$sn\ \xi = \sin\theta = \sin\xi,$$
$$cn\ \xi = \cos\theta = \cos\xi,$$
$$dn\ \xi = 1,$$

and the solution (**7·07**) reduces, as of course it should, to that already found for this special case (equations **7·056** and **7·057**).

§ 7·1.   Geometrical Exposition

We have in (**7·07**) the solution of the problem of the motion of a rigid body, one point in which is fixed, for the special case where the forces acting on the body have no resultant moment about the fixed point. A very instructive picture of the motion is provided by the geometrical method of Poinsot (*Théorie nouvelle de la rotation des corps*, 1851). The results we have already obtained indicate that the instantaneous axis of rotation wanders about in the rigid body and therefore sweeps out in it a cone (*s*, Fig. 7·1), having its apex at the fixed point, O. The positions of this axis *in the body* at successive instants of time are represented by O*a*, O*b*, O*c*, O*d*, O*e*, etc. The lengths of these lines may conveniently be made equal, or proportional, to the corresponding values of **ω** at these instants. During the time required by the axis of rotation to travel from O*a* to O*b* the point *b* will travel in space to some point β. That is to say, the line O*b* in the body will occupy the position Oβ at the instant when it coincides with the axis of rotation. In a succeeding interval the axis of rotation will have reached O*c*, (in the body) which

will now have a position Oγ in space, and so forth.   The lines,
Oα, Oβ, Oγ, Oδ, Oε, etc., sweep out a cone, σ, which is fixed *in
space*.   The motion of the body
is consequently such as would
result if a certain cone, s, fixed
rigidly in the body, were to *roll*,
with an appropriate angular
velocity, on another cone, σ,
fixed in space.   The cone, s, will
cut the momental ellipsoid (which
may likewise be described as
fixed in the body, or rigidly
attached to it) in a closed curve,
as will be shown.   This curve
Poinsot called the **polhode**
($\pi\acute{o}\lambda o\varsigma$, axis ; $\acute{o}\delta\acute{o}\varsigma$, path).   Its
equations can be found in the
following way : Using $(x,\ y,\ z)$
in place of $(\xi,\ \eta,\ \zeta)$ in the equa-

FIG. 7·1

tion (**6·235**), of the momental ellipsoid, we have for the com-
ponents of the angular velocity, $\boldsymbol{\omega}$,
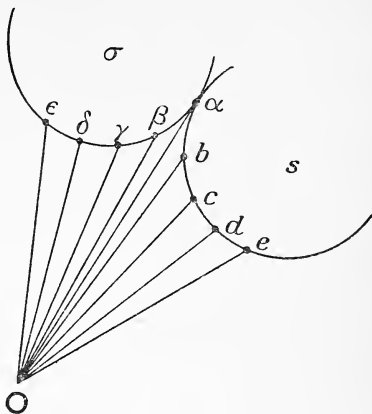
$$\omega_x = \frac{\omega}{\varrho}x,\ \ \omega_y = \frac{\omega}{\varrho}y,\ \ \omega_z = \frac{\omega}{\varrho}z\ \ \ .\ \ \ .\ \ \ .\ \ \ (7\text{·}1)$$

ϱ meaning, as in § 6·2, the radius vector from O to $(x,\ y,\ z)$.
The perpendicular, p, from O to the tangent plane at $(x,\ y,\ z)$ is

$$p = \varrho\cos\theta,$$

if θ is the angle between **p** and $\boldsymbol{\varrho}$.   Therefore p is the scalar
product, $(\boldsymbol{\varrho}\,\mathbf{N})$, of $\boldsymbol{\varrho}$ and a unit vector **N** in the direction **p**.
Consequently

$$p = x\alpha + y\beta + z\gamma,$$

if α, β and γ are the components of **N**, or the direction cosines of **p**.
The equation of the tangent plane at $(x,\ y,\ z)$ is

$$Ax\xi + By\eta + Cz\zeta = M,$$

if $(\xi,\ \eta,\ \zeta)$ is any point on it.   Therefore

$$\alpha = \frac{Ax}{\sqrt{A^2x^2 + B^2y^2 + C^2z^2}|}\ ,$$

$$\beta = \frac{By}{\sqrt{A^2x^2 + B^2y^2 + C^2z^2}|}\ ,$$

$$\gamma = \frac{Cz}{\sqrt{A^2x^2 + B^2y^2 + C^2z^2}|}\ ,$$

$$p = \frac{M}{\sqrt{A^2x^2 + B^2y^2 + C^2z^2}|}\ \ \ .\ \ \ .\ \ \ .\ \ \ (7\text{·}11)$$

But by combining (7·04) and (7·1), we find that

$$A^2x^2 + B^2y^2 + C^2z^2 = \frac{\Omega^2\rho^2}{\omega^2} \quad . \quad . \quad . \quad (7\cdot12)$$

Therefore

$$p = \frac{M\omega}{\Omega\rho} \quad . \quad . \quad . \quad . \quad (7\cdot121)$$

Similarly, by combining (6·41) and (7·1) we obtain

$$Ax^2 + By^2 + Cz^2 = 2T\frac{\rho^2}{\omega^2} \quad . \quad . \quad . \quad (7\cdot13)$$

or

$$M = 2T\frac{\rho^2}{\omega^2} \quad . \quad . \quad . \quad (7\cdot131)$$

It follows that $\dfrac{\omega}{\rho}$ is a constant, namely

$$\frac{\omega}{\rho} = \sqrt{\frac{2T}{M}} \bigg| \quad . \quad . \quad . \quad . \quad (7\cdot132)$$

and consequently

$$p = \frac{\sqrt{2TM}|}{\Omega} \quad . \quad . \quad . \quad . \quad (7\cdot133)$$

It is therefore constant and its direction cosines (7·11) are the same as those of the angular momentum Ω. Consequently, it is invariable in length and direction, and the tangent plane remains fixed in space during the motion of the body.

The last of the equations (7·11) gives us

$$A^2x^2 + B^2y^2 + C^2z^2 = \frac{M^2}{p^2} \quad . \quad . \quad . \quad (7\cdot14)$$

This equation holds for any point (x, y, z) where the axis of rotation cuts the ellipsoid of inertia and it, together with the equation of the ellipsoid,

$$Ax^2 + By^2 + Cz^2 = M \quad . \quad . \quad . \quad (7\cdot15)$$

determines the polhode.

If we multiply (7·14) by p^2 and (7·15) by M and subtract, we get

$$(p^2A^2 - MA)x^2 + (p^2B^2 - MB)y^2 + (p^2C^2 - MC)z^2 = 0 \quad (7\cdot16)$$

which is the equation of the polhode cone s.

The curve traced out on the fixed tangent plane by the instantaneous axis of rotation was called by Poinsot the **herpolhode** (from ἕρπειν, to crawl, like a serpent). The corresponding herpolhode cone is the cone σ, fixed in space, on which the polhode cone rolls. We have now a very clear picture of the motion, especially if we remember (7·132) that the angular velocity about the instantaneous axis is proportional to

ρ, the radius vector of the momental ellipsoid which coincides with the axis. The cone, s, fixed relatively to the ellipsoid rolls on the cone, σ, in such a way that the ellipsoid is in contact with a fixed plane, the velocity of rotation at any instant being proportional to the distance, ϱ, from the fixed point, O, to the point of contact with the fixed plane.

The semi-axes of the ellipsoid of inertia are

$$\sqrt{\frac{M}{A}}, \ \sqrt{\frac{M}{B}}, \ \sqrt{\frac{M}{C}}.$$

Suppose,

$$A > B > C,$$

then

$$\frac{M}{A} \leqq p^2 \leqq \frac{M}{C} \quad \cdots \cdots \quad (7\cdot17)$$

In one extreme case

$$p^2 = \frac{M}{A}$$

and the equation of the polhode cone s (7·16) becomes

$$(B^2 - AB)y^2 + (C^2 - AC)z^2 = 0.$$

Since both terms on the left of this equation have the same sign, the only real points on it are the points $y = z = 0$, and the cone reduces to a straight line, or, strictly speaking, to two imaginary planes intersecting in a real line, the X axis. There is a similar state of affairs if p^2 has the other extreme value $\dfrac{M}{C}$. If however

$$p^2 = \frac{M}{B},$$

the equation of the cone becomes

$$(A^2 - AB)x^2 + (C^2 - CB)z^2 = 0.$$

In this equation $A^2 - AB$ is positive and $C^2 - CB$ is negative. It therefore represents two *real* planes intersecting in the Y axis.

Instead of combining equations (7·14) and (7·15) to get the equation of a cone, let us eliminate x^2. We thus obtain

$$(B^2 - AB)y^2 + (C^2 - AC)z^2 = \frac{M^2}{p^2} - AM \quad . \quad (7\cdot18)$$

Reference to (7·17) will show that the right-hand member of this equation is negative or, in the extreme case, zero, and since this is true likewise of the coefficients of y^2 and z^2, we conclude that the projections of the polhodes on the YZ plane are ellipses.

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The ratio of the semi-axes of *any* of the ellipses is

$$\sqrt{\frac{B(A-B)}{C(A-C)}} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \textbf{(7·181)}$$

Similarly, we can show that the projections of the polhodes on the XY plane are the ellipses,

$$(A^2 - AC)x^2 + (B^2 - BC)y^2 = \frac{M^2}{p^2} - MC \quad \textbf{(7·182)}$$

and ratio of the semi-axes being in this case

$$\sqrt{\frac{A(C-A)}{B(C-B)}} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \textbf{(7·183)}$$

This result should be compared with (**7·057**).

The projections on the XZ plane are the hyperbolas

$$(A^2 - AB)x^2 + (C^2 - CB)z^2 = \frac{M^2}{p^2} - MB \quad \textbf{(7·184)}$$

§ 7·2. Euler's Angular Co-ordinates

We shall continue to use a system of axes, X', Y', Z', fixed in the body, and coincident with its principal axes of inertia.
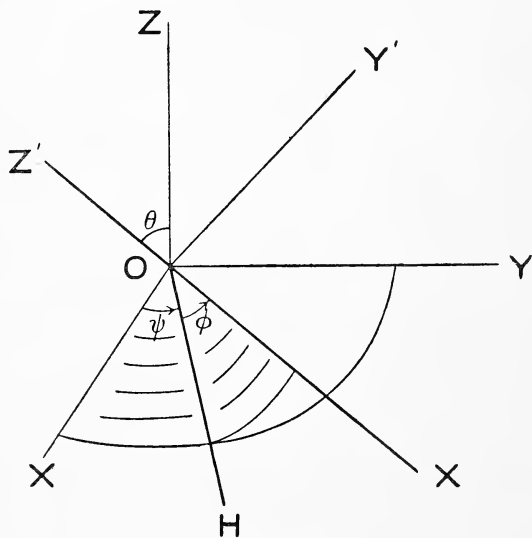

Fig. 7·2

Let X, Y, Z, be another set of axes fixed in space, the Z axis being directed vertically upwards, and the two sets of axes having a common origin, O, in the fixed point of the body.   Let the angle

between Z and Z' be denoted by θ. The $X'Y'$ plane intersects the XY plane in the line, OH, (Fig. 7·2). The angle between OH and OX is denoted by ψ, and that between OX' and OH by ϕ. The positive directions are indicated in the figure by arrows. The position of the body, at any instant, is completely determined by the values of these three angles, called **Euler's angles.**

The Eulerian angles are illustrated by the method of mounting an ordinary gyroscope (Fig. 7·21). There is a fixed ring, ABC. Within this is a second ring abc pivoted at A and B so that it can turn about the vertical axis, AB. The axis AB corresponds to OZ (Fig. 7·2). Within the ring, abc, is still another ring, $\alpha\beta\gamma$, pivoted at a and b, so that it can turn about the horizontal axis, ab. This axis corresponds to OH (Fig. 7·2). The gyroscopic wheel, itself, is pivoted at α and β in the innermost ring, so that it can spin about an axis, $\alpha\beta$, perpendicular to ab. The axis, $\alpha\beta$, corresponds to OZ'.
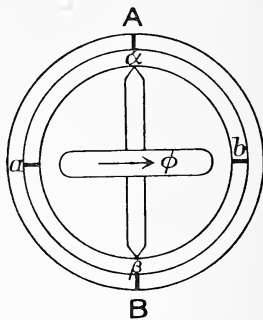
FIG. 7·21

Let us now express the components, ω_x, ω_y and ω_z,[1] of the angular velocity of the body, in terms of $\dfrac{d\psi}{dt}$, $\dfrac{d\theta}{dt}$ and $\dfrac{d\phi}{dt}$. It is clear that ω_x and ω_y do not depend on $\dfrac{d\phi}{dt}$ and we must therefore have

$$\omega_x = \frac{d\psi}{dt} \cos (ZX') + \frac{d\theta}{dt} \cos (HX'),$$

$$\omega_y = \frac{d\psi}{dt} \cos (ZY') + \frac{d\theta}{dt} \cos (HY').$$

Obviously ω_z is not identical with $\dfrac{d\phi}{dt}$ since ϕ is an angle measured from the moving line OH. To get ω_z we have to add to $\dfrac{d\phi}{dt}$ the angular $\dfrac{d\psi}{dt}$ multiplied by $\cos (ZZ')$, therefore

$$\omega_z = \frac{d\phi}{dt} + \frac{d\psi}{dt} \cos (ZZ').$$

[1] Note that ω_x, ω_y, ω_z have the same meaning as in Euler's equations. They are the components of the angular velocity referred to axes X', Y', Z' fixed in the body.

The direction cosines in these equations are easily seen to have the values set out in the table :

	X'	Y'	Z'
Z	$\sin \theta \sin \phi$	$\sin \theta \cos \phi$	$\cos \theta$
H	$\cos \phi$	$- \sin \phi$	0

For example, cos (ZX') is X' co-ordinate of a point on Z the unit distance from O. The distance from O of the projection of this point on the $X'Y'$ plane is $\sin \theta$ and the angle between this projection and OX' is obviously the complement of ϕ. Hence we get the projection on OX' by a further multiplication by $\sin \phi$.

We therefore arrive at the following relationships : —

$$\omega_x = \frac{d\psi}{dt} \sin \theta \sin \phi + \frac{d\theta}{dt} \cos \phi,$$

$$\omega_y = \frac{d\psi}{dt} \sin \theta \cos \phi - \frac{d\theta}{dt} \sin \phi,$$

$$\omega_z = \frac{d\psi}{dt} \cos \theta + \frac{d\phi}{dt}. \quad . \quad . \quad . \quad . \quad . \quad \textbf{(7·2)}$$

§ 7·3. The Top and Gyroscope

We shall now apply Euler's equations to the problem of the symmetrical top (or gyroscope) supposing the peg of the top (or the fixed point in the gyroscope) to be fixed in the origin. If the Z' axis is the axis of symmetry of the top, and if the distance of the centre of mass from O is h, the couple exerted has always the direction OH, and is equal to $mgh \sin \theta$, m being the mass of the top. We must substitute for L, M and N in Euler's equations the components of this couple along the directions X', Y' and Z'. The table of cosines (§ 7·2) gives us

$$L = mgh \sin \theta \cos \phi,$$
$$M = -mgh \sin \theta \sin \phi,$$
$$N = 0.$$

On substituting these values for L, M and N in Euler's equations, (7·02), we have

$$mgh \sin \theta \cos \phi = A\frac{d\omega_x}{dt} + (C - B)\omega_y\omega_y,$$

$$- mgh \sin \theta \sin \phi = B\frac{d\omega_y}{dt} + (A - C)\omega_z\omega_x,$$

$$0 = C\frac{d\omega_z}{dt} + (B - A)\omega_x\omega_y. \quad (7·3)$$

If we replace ω_x, ω_y and ω_z in (7·3) by the Eulerian expressions (7·2) we obtain three differential equations the solution of which gives the character of the motion of the top. Instead of proceeding in this way it is simpler to make use of the energy equation, and obtain two further equations by equating the angular momenta about the Z and Z' axes to constants. This we are at liberty to do, since the applied couple is in the direction OH, that is to say, in a direction perpendicular to Z and to Z', so that its component in either of these directions is zero.

We obtain the energy equation by multiplying equations (7·3) by ω_x, ω_y, and ω_z respectively and adding. In this way we get

$$\frac{d}{dt}\{\tfrac{1}{2}A\omega_x{}^2 + \tfrac{1}{2}B\omega_y{}^2 + \tfrac{1}{2}C\omega_z{}^2\} = mgh \sin \theta(\omega_x \cos \phi - \omega_y \sin \phi).$$

If now we write $A = B$, on account of the symmetry, and substitute for ω_x and ω_y their Eulerian values (7·2), we have

$$\frac{d}{dt}\left\{\tfrac{1}{2}A\left[\sin^2 \theta\left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2\right] + \tfrac{1}{2}C\omega_0{}^2\right\} = mgh \sin \theta.\frac{d\theta}{dt},$$

in which we have replaced $\omega_z{}^2$ by $\omega_0{}^2$, which is a constant by the third of the equations (7·3). Thus on integrating we arrive at the result

$$\sin^2 \theta\left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2 = a - \frac{2mgh}{A} \cos \theta \quad . \quad (7·31)$$

where a is a constant of integration. This is the energy equation.

The table of direction cosines (§ 7·2) gives for the angular momentum in the Z direction,

$$A\omega_x \sin \theta \sin \phi + B\omega_y \sin \theta \cos \phi + C\omega_z \cos \theta,$$

or $\quad A \sin \theta(\omega_x \sin \phi + \omega_y \cos \phi) + C\omega_0 \cos \theta.$

On replacing ω_x and ω_y in the usual way by the expressions in (7·2), we get for the angular momentum about the Z axis,

$$A \sin \theta\left(\sin \theta \frac{d\psi}{dt}\right) + C\omega_0 \cos \theta = \text{a constant},$$

or $\quad \sin^2 \theta \frac{d\psi}{dt} = a - \frac{C\omega_0}{A} \cos \theta . \quad . \quad . \quad . \quad (7·32)$

where a is a constant of integration.

For the third equation we have

$$C\omega_z = \text{a constant,}$$

or, by (7·2),

$$C\left\{\cos\theta\,\frac{d\psi}{dt} + \frac{d\phi}{dt}\right\} = \text{a constant,}$$

or finally

$$\cos\theta\frac{d\psi}{dt} + \frac{d\phi}{dt} = \omega_0 \qquad . \quad . \quad . \quad . \quad (7\cdot33)$$

The three equations, 7·31, 7·32 and 7·33 completely describe the behaviour of the ordinary top, when its peg is prevented from wandering about, or the motion of any rigid body with axial symmetry (gyroscope), when one point on the axis is fixed in space. By eliminating $\dfrac{d\psi}{dt}$ from (7·31) and (7·32) we arrive at the equation

$$\frac{1}{\sin^2\theta}\left\{a - \frac{C\omega_0}{A}\cos\theta\right\}^2 + \left(\frac{d\theta}{dt}\right)^2 = a - \frac{2mgh}{A}\cos\theta \quad (7\cdot34)$$

We can simplify this and the remaining equations by the following abbreviations :

$$\beta = \frac{2mgh}{A}, \; b = \frac{C\omega_0}{A}, \; \mu \; = \cos\theta \quad . \quad . \quad (7\cdot345)$$

and consequently

$$-\sin\theta\frac{d\theta}{dt} = \frac{d\mu}{dt} \quad . \quad . \quad . \quad . \quad (7\cdot346)$$

We have therefore, when we substitute in (7·34),

$$\left(\frac{d\mu}{dt}\right)^2 = (a - \beta\mu)(1 - \mu^2) - (a - b\mu)^2 \quad . \quad (7\cdot35)$$

an equation which may be expressed in the integrated form

$$t = \int_c^\mu \frac{d\mu}{\sqrt{(a - \beta\mu)(1 - \mu^2) - (a - b\mu)^2}} \qquad (7\cdot351)$$

if c is the value of μ at the instant $t = 0$.

Equations (7·32) and (7·33) take the respective forms,

$$\frac{d\psi}{dt} = \frac{a - b\mu}{1 - \mu^2} \qquad . \quad . \quad . \quad . \quad . \quad (7\cdot36)$$

$$\frac{d\phi}{dt} = \omega_0 - \frac{\mu(a - b\mu)}{1 - \mu^2} \qquad . \quad . \quad . \quad (7\cdot37)$$

The integral (7·351) belongs to the class of elliptic integrals, and therefore μ, or $\cos\theta$, is an elliptic function of the time and

consequently oscillates periodically between a fixed upper limit μ_0, and a fixed lower limit μ_1. Otherwise expressed, the angle, θ, between the axis of symmetry and the vertical, will change in a periodic fashion between a smallest value θ_0 and a greatest value θ_1. This motion of the axis is called **nutation**. The motion expressed by $\dfrac{d\psi}{dt}$, that is to say, the motion of the axis, OH (Fig. 7·2), is called **precession**.

We can easily learn the general character of the motion from equations (**7·35**), (**7·36**) and (**7·37**) without making explicit use of the properties of elliptic functions. If we denote $\left(\dfrac{d\mu}{dt}\right)^2$ by $f(\mu)$, equation (**7·35**) becomes

$$f(\mu) = (\alpha - \beta\mu)(1 - \mu^2) - (a - b\mu)^2.$$

Since β is a positive constant,

$$f(-\infty) = -\infty,$$
$$f(+\infty) = +\infty,$$

and further

$$f(-1) = -(a + b)^2,$$
$$f(+1) = -(a - b)^2.$$

Therefore $f(-1)$ and $f(+1)$ are necessarily negative (or zero), and the general character of the function $f(\mu)$ $\left(= \left(\dfrac{d\mu}{dt}\right)^2\right)$ is that illustrated by Fig. 7·3. Only *positive* values of $f(\mu)$ and values of μ between -1 and $+1$ can have any significance in the motion of the top. The significant points in the diagrams (Fig. 7·3) are therefore those in the shaded areas of a, b and c. During the motion μ varies backwards and forwards between fixed upper and lower limits, μ_0 and μ_1 respectively and associated with this is a corresponding variation of the angle, θ. At the same time the precessional velocity, $\dfrac{d\psi}{dt}$, will also vary periodically with the same period as μ (see equation (**7·36**)). If we restrict our attention to the case where a and b are positive, we have the following possibilities : (1) if b is small enough, i.e. if the top is not spinning fast (see (**7·345**)), $\dfrac{d\psi}{dt}$ will remain positive (**7·36**) ;

(2) if the top is spinning very fast (b large enough) $\dfrac{d\psi}{dt}$ may change sign during the nutational motion between θ_1 and θ_0. This will happen when μ is equal to $\dfrac{a}{b}$. A special case, (3), is that in

7

which $\mu_0 = \dfrac{a}{b}$. This is the case when the top is set spinning

and released in such a way that initially $\dfrac{d\theta}{dt} = 0$ and $\dfrac{d\psi}{dt} = 0$.
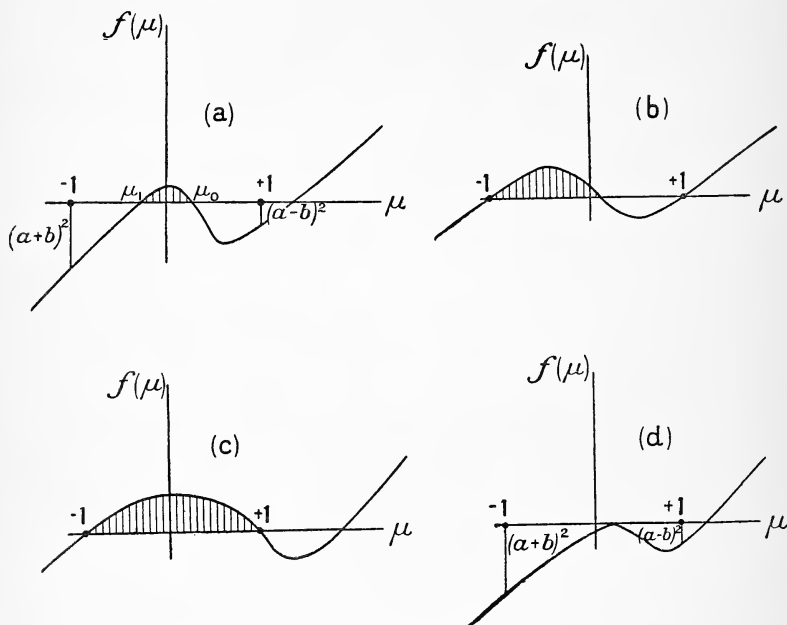
Fig. 7·3

The three cases considered are illustrated by (a), (b) and (c) respectively in Fig. 7·31, which exhibits the curve traced out by the centre of mass on the sphere with centre, O, and radius h.
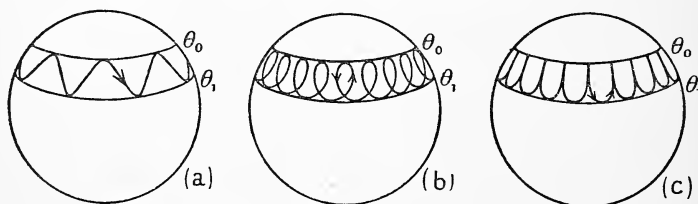

Fig. 7·31

The figure illustrates the possibilities in the case of the *ordinary top*, for which μ is always positive.

It is instructive to consider the case where the top is not spinning and where the angular momentum about the vertical

axis, Z, is zero, so that ω_0 and a are both zero, (7·345) and (7·32). We have then the equation

$$f(\mu) = \left(\frac{d\mu}{dt}\right)^2 = (a - \beta\mu)(1 - \mu^2),$$

or if we substitute $\cos\theta$ for μ,

$$\left(\frac{d\theta}{dt}\right)^2 = (a - \beta\cos\theta).$$

As $\beta = \dfrac{2mgh}{A}$ we may express the equation in the form

$$\tfrac{1}{2}\left(\frac{d\theta}{dt}\right)^2 + \frac{mgh}{A}\cos\theta = \frac{a}{2}.$$

This equation is seen to be identical with (6·501), since A and I have the same meaning, and the difference in sign is merely due to the fact that θ in the one equation is the supplement of θ in the other. The motion is that of the pendulum. We find, just as in § 6·5, two sub-cases. If $\dfrac{a}{2}$ exceeds $\dfrac{mgh}{A}$, i.e. if a exceeds β, $\dfrac{d\theta}{dt}$ never vanishes and body rotates continuously round a fixed horizontal axis, but with a periodically varying velocity. This is the case illustrated by (c) in Fig. 7·3, since $\left(\dfrac{d\mu}{dt}\right)^2 = f(\mu)$ vanishes at the points $\mu = +1$ and $\mu = -1$ and nowhere else between these limits. On the other hand, if $a < \beta$, $f(\mu)$ again vanishes at $+1$ and -1, (7·35) since a and b are both zero, *but also at a point*, $\mu = \dfrac{a}{\beta}$, *between these limits*. This corresponds to the ordinary pendulum motion and is illustrated in (b), Fig. 7·3. The significant values of μ extend from -1 (when the pendulum is vertical) to $\dfrac{a}{\beta}$.

Another interesting special case is that for which the interval $\mu_0\mu_1$ within which $f(\mu)$ is positive is contracted to a point, so that the curve for $f(\mu)$ touches the μ axis as in Fig. 7·3 (d). Therefore μ is constant during the motion and consequently so is $\dfrac{d\psi}{dt}$. The axis of the top or gyroscope sweeps out a circular cone in space with a constant angular velocity. Let us consider the case where μ is zero and the axis of symmetry therefore horizontal. We see (7·36) that $\dfrac{d\psi}{dt}$ is equal to the constant a. It is an instructive exercise to determine a by means of equations (7·01). In

these equations we must remember that $(\omega_x, \omega_y, \omega_z)$ represents the angular velocity of the *moving axes* and not necessarily that of the gyroscope. Suppose the X' directed vertically upwards, Fig. 7·32, and *fixed*, while the axis of symmetry of the top or gyroscope coincides with the Z' axis. We have therefore for $(\omega_x, \omega_y, \omega_z)$ of equations (**7·01**)

$$\omega_x = a,$$
$$\omega_y = 0,$$
$$\omega_z = 0.$$

FIG. 7·32

Let $P_x{}', P_y{}'$ and $P_z{}'$ be the components of angular momentum relative to these axes. Then

$$P_x{}' = 0,$$
$$P_y{}' = 0,$$
$$P_z{}' = C\omega_0,$$

where ω_0 has the same meaning as before. We have further

$$\frac{dP_x}{dt} = 0,$$

$$\frac{dP_y}{dt} = -mgh,$$

$$\frac{dP_z}{dt} = 0,$$

since these quantities represent the rate of change of angular momentum with respect to the fixed axes, which are momentarily coincident with the moving ones. The equations (**7·01**) are then satisfied if

$$-mgh = -C\omega_0 a,$$

or

$$a = \frac{mgh}{C\omega_0}.$$

We can of course arrive at this result in a much simpler way. Let OA (Fig. 7·33) represent the angular momentum $C\omega_0$ at any instant. The applied couple will produce in a short interval dt a change of momentum $d\Omega$, at right angles to OA, as shown in the figure. The angle $d\psi$ swept out during dt will therefore be

FIG. 7·33

$$\frac{d\Omega}{C\omega_0} = d\psi$$
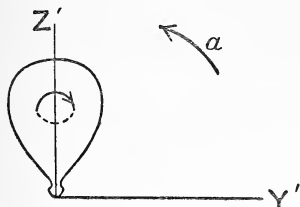
or
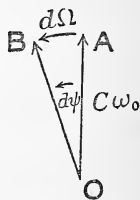
$$\frac{\dfrac{d\Omega}{dt}}{C\omega_0} = \frac{d\psi}{dt}.$$

But in the present case $\dfrac{d\Omega}{dt}$ is constant and equal to mgh, therefore

$$\frac{d\psi}{dt} = a = \frac{mgh}{C\omega_0},$$

as we have already found.

§ 7·4. THE PRECESSION OF THE EQUINOXES

The earth behaves like a top. The attraction of the sun is exerted along a line which does not pass through the centre of mass of the earth except at the equinoxes. It thus gives rise to a couple tending to tilt the earth's axis about its centre of mass and make it more nearly vertical. The state of affairs is very similar to that we have just been studying. The centre of mass of the earth corresponds to the fixed point, O, the peg of top. Consequently the earth's axis exhibits a motion of nutation and precession. The line in which the equator cuts the ecliptic corresponds to the OH in Fig. 7·2. The points where it cuts the celestial sphere are called the **equinoctial points,** from the circumstance that day and night are equal in length when the sun passes through them. In consequence of the precession the equinoctial points travel slowly round the heavens in the plane of the ecliptic in a retrograde direction, a whole revolution requiring a period of 25,800 years. Associated with this is a corresponding motion of the celestial poles which in the same period describe circles of 23° 27′ in diameter round the poles of the ecliptic.

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CHAPTER V

PRINCIPLES OF DYNAMICS

§ 8. PRINCIPLE OF VIRTUAL DISPLACEMENTS

IN order that a particle may be in equilibrium, the resultant of all the forces acting on it must necessarily be zero. If $\mathbf{F} \equiv (F_x, F_y, F_z)$ be the resultant force,

$$F_x = F_y = F_z = 0.$$

This condition may be stated in the following alternative way :

$$(\mathbf{F} \, \delta\mathbf{l}) = F_x\delta x + F_y\delta y + F_z\delta z = 0 \quad . \quad . \quad (8)$$

where $\delta\mathbf{l} \equiv (\delta x, \delta y, \delta z)$ is an *arbitrary* small displacement of the particle, i.e. any small displacement we like to choose. For suppose we assign to δy and δz the value zero, and to δx a value different from zero. Equation (8) then becomes

$$F_x\delta x = 0,$$

and hence $F_x = 0$. Similarly the statement (8) requires F_y and F_z to be zero.

Consider any number of particles, which we may distinguish by the subscripts 1, 2, 3, . . . s, . . . , and let the respective forces acting on them be $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots \mathbf{F}_s \ldots$ Further, imagine the particles to suffer the arbitrary small displacements, $(\delta x_1, \delta y_1, \delta z_1)$, $(\delta x_2, \delta y_2, \delta z_2)$, . . . $(\delta x_s, \delta y_s, \delta z_s)$, . . . Then the condition for the equilibrium of the system of particles is

$$\sum_s (F_{sx}\delta x_s + F_{sy}\delta y_s + F_{sz}\delta z_s) = 0 \quad . \quad . \quad (8\cdot01)$$

the summation being extended over all the particles of the system. The arbitrary small displacement $(\delta x, \delta y, \delta z)$ is called a **virtual displacement** and the statement (8) or (8·01) is called the **principle of virtual displacements** or the **principle of virtual work.**

The utility of the principle becomes evident when we apply it to cases where the particles are subject to **constraints.** As an illustration consider the case of a single particle so constrained that it cannot leave some given surface. There will in general be some force, $\mathbf{F}' \equiv (F_x', F_y', F_z')$, *normal* to the surface, and

of such a magnitude that it prevents the particle from leaving it.
Let us write equation (8) in the form

$$(F_x + F_x{}')\delta x + (F_y + F_y{}')\delta y + (F_z + F_z{}')\delta z = 0,$$

where $\mathbf{F} \equiv (F_x, F_y, F_z)$ represents the part of the force on the
particle not due to the constraint. We shall call it the **impressed
force.** Of course the principle of virtual displacements requires
that

$$F_x + F_x{}' = F_y + F_y{}' = F_z + F_z{}' = 0 \; ;$$

but this is not the most important, nor the most interesting
inference from the equation. If we subject the virtual dis-
placement $\delta \mathbf{l} \equiv (\delta x, \delta y, \delta z)$ to the condition that it has to be
along the surface, we have, since $\mathbf{F}'$ is normal to the surface,

$$(\mathbf{F}'\delta \mathbf{l}) = F_x{}'\delta x + F_y{}'\delta y + F_z{}'\delta z = 0$$

and consequently

$$F_x\delta x + F_y\delta y + F_z\delta z = 0 \quad . \quad . \quad . \quad (8\cdot02)$$

In this statement of the principle all reference to the force $\mathbf{F}'$
due to the constraint is eliminated ; but in applying it we have to
remember that the virtual displacement *is no longer arbitrary*,
and we cannot infer therefore that $F_x = F_y = F_z = 0$. Indeed,
this would in general be untrue. Let the equation of the surface,
to which any motion of the particle is confined, be

$$\phi \, (x, y, z) = 0 \quad . \quad . \quad . \quad . \quad (8\cdot03)$$

The virtual displacement is therefore subject to the condition

$$\frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z = 0 . \quad . \quad . \quad (8\cdot031)$$

Let us eliminate one of the components of $\delta \mathbf{l}$, e.g., δx, with the
help of (8·02). We can do this most conveniently by multi-
plying (8·031) by a factor, λ, so chosen that

$$F_x - \lambda\frac{\partial \phi}{\partial x} = 0 \quad . \quad . \quad . \quad . \quad (8\cdot032)$$

and subtracting the result from (8·02). This gives

$$\left(F_y - \lambda\frac{\partial \phi}{\partial y}\right)\delta y + \left(F_z - \lambda\frac{\partial \phi}{\partial z}\right)\delta z = 0 . \quad (8\cdot033)$$

The components δy and δz of the virtual displacement can be
chosen arbitrarily, since whatever small values we assign to them
we can always so adjust δx as to satisfy (8·031), the condition
to which the displacement has to conform. Hence we infer

$$F_y - \lambda\frac{\partial \phi}{\partial y} = 0,$$

$$F_z - \lambda\frac{\partial \phi}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad (8\cdot034)$$

In order therefore that the particle may be in equilibrium the impressed force, $\mathbf{F} \equiv (F_x, F_y, F_z)$ must satisfy the equations (8·032) and (8·034), or, what amounts to the same thing,

$$\frac{F_x}{\dfrac{\partial \phi}{\partial x}} = \frac{F_y}{\dfrac{\partial \phi}{\partial y}} = \frac{F_z}{\dfrac{\partial \phi}{\partial z}} \quad \cdots \quad (8\cdot035)$$

Consider next the case where a particle is constrained to keep to a curve. Suppose the latter to be the intersection of two surfaces,

$$\phi\,(x, y, z) = 0,$$
$$\psi\,(x, y, z) = 0 \quad \cdots \quad (8\cdot04)$$

The virtual displacement, $\delta\mathbf{l} \equiv (\delta x, \delta y, \delta z)$ has consequently to satisfy the conditions

$$\frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z = 0,$$

$$\frac{\partial \psi}{\partial x}\delta x + \frac{\partial \psi}{\partial y}\delta y + \frac{\partial \psi}{\partial z}\delta z = 0 \quad \cdots \quad (8\cdot041)$$

and we infer that

$$F_x - \lambda\frac{\partial \phi}{\partial x} - \mu\frac{\partial \psi}{\partial x} = 0,$$

$$F_y - \lambda\frac{\partial \phi}{\partial y} - \mu\frac{\partial \psi}{\partial y} = 0,$$

$$F_z - \lambda\frac{\partial \phi}{\partial z} - \mu\frac{\partial \psi}{\partial z} = 0, \quad \cdots \quad (8\cdot05)$$

or, what amounts to the same thing,

$$\begin{vmatrix} F_x, & \dfrac{\partial \phi}{\partial x}, & \dfrac{\partial \psi}{\partial x} \\[2ex] F_y, & \dfrac{\partial \phi}{\partial y}, & \dfrac{\partial \psi}{\partial y} \\[2ex] F_z, & \dfrac{\partial \phi}{\partial z}, & \dfrac{\partial \psi}{\partial z} \end{vmatrix} = 0 \quad \cdots \quad (8\cdot051)$$

The principle of virtual displacements may be illustrated by the following examples :—

Let the particle be confined to a spherical surface, but otherwise perfectly free, and suppose the force impressed on it to be directed vertically downwards. It might, for instance, be its weight. Let the origin of co-ordinates be at the centre of the

sphere and the X axis have the direction of the force.   Equation (**8·031**) becomes

$$x\delta x + y\delta y + z\delta z = 0,$$

and for (**8·032**) and (**8·034**) we have

$$F_x = \lambda x,$$
$$F_y = \lambda y,$$
$$F_z = \lambda z.$$

Now since $F_y = F_z = 0$,

$$F_x = \lambda x,$$
$$0 = \lambda y,$$
$$0 = \lambda z,$$

and as F_x is not zero, λ cannot vanish, and therefore

$$y = z = 0.$$

Consequently $x = + r$ or $- r$, where r is the radius of the sphere, or the possible positions of equilibrium are the uppermost and lowermost points on the sphere.[1]

An instructive example is that of a rigid body which can turn freely about a fixed axis, which we shall take to be the Z axis of rectangular co-ordinates.   The conditions for equilibrium are expressed by equations (**8·01**), the forces, **F**, being the impressed forces : not those due to constraints, together with the equations describing the constraints.   These latter are, for every particle, s,

$$\delta z_s = 0,$$
$$\delta x_s = - y_s\delta\phi,$$
$$\delta y_s = x_s\delta\phi,$$

where $\delta\phi$ is the same for all the particles in the body.   Equation (**8·01**) therefore becomes

$$\delta\phi\underset{s}{\Sigma}(x_sF_{sy} - y_sF_{sx}) = 0.$$

Now $\delta\phi$ is arbitrary, hence

$$\underset{s}{\Sigma}(x_sF_{sy} - y_sF_{sx}) = 0.$$

This means that the sum of the moments of all the impressed forces with respect to the Z axis is zero, a result we have already obtained by a different method (§ 6·1).

§ 8·1. Principle of d'Alembert

The principle of virtual displacements is a statical one.   It provides a means of investigating the conditions necessary for

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[1] Alternatively, the equations (**8·035**) become $\dfrac{F_x}{x} = \dfrac{F_y}{y} = \dfrac{F_z}{z}$ and as $F_y = F_z = 0$ it follows that $y = z = 0$.

the *equilibrium* of a dynamical system. Its scope can be extended, however, by a device due to d'Alembert (*Traité de Dynamique*, 1743), so as to furnish a wider principle which constitutes a basis for the general investigation of the behaviour of dynamical systems.

Let $\mathbf{F} \equiv (F_x, F_y, F_z)$ be the resultant force exerted on a particle of mass m, then

$$m\frac{d^2x}{dt^2} - F_x = 0,$$

$$m\frac{d^2y}{dt^2} - F_y = 0,$$

$$m\frac{d^2z}{dt^2} - F_z = 0.$$

We may express these equations in the single statement

$$\left(m\frac{d^2x}{dt^2} - F_x\right)\delta x + \left(m\frac{d^2y}{dt^2} - F_y\right)\delta y + \left(m\frac{d^2z}{dt^2} - F_z\right)\delta z = 0 \quad \textbf{(8·1)}$$

if $(\delta x, \delta y, \delta z)$ is an arbitrary small displacement, since this necessitates the vanishing of the coefficients of δx, δy and δz. Now **(8·1)** can be extended to apply to a system of particles, subject possibly to constraints, in the following way :

$$\sum_s \left\{ \left(m_s\frac{d^2x_s}{dt^2} - F_{sx}\right)\delta x_s + \left(m_s\frac{d^2y_s}{dt^2} - F_{sy}\right)\delta y_s \right.$$

$$\left. + \left(m_s\frac{d^2z_s}{dt^2} - F_{sz}\right)\delta z_s \right\} = 0 \quad . \quad . \quad \textbf{(8·11)}$$

If the system should be subject to constraints, $\mathbf{F}_s$ will signify the force *impressed* on the particle, s, and will not include the force or forces due to the constraints, and the virtual displacements $(\delta x_s, \delta y_s, \delta z_s)$ are not all arbitrary, but subject to the equations defining the constraints.

Equation **(8·11)**, with the interpretation just given, expresses the **principle of d'Alembert.**

A simple illustration of the principle is furnished by the example of the rigid body in § 8. The procedure here differs only in the substitution of $m\frac{d^2x_s}{dt^2} - F_{sx}$ for the F_{sx} of § 8, and corresponding expressions for F_{sy} and F_{sz}. We thus get

$$\delta\phi\sum_s\left\{x_s\left(m_s\frac{d^2y_s}{dt^2} - F_{sy}\right) - y_s\left(m_s\frac{d^2x_s}{dt^2} - F_{sx}\right)\right\} = 0,$$

and therefore, on account of the arbitrariness of $\delta\phi$,

$$\sum_s \left\{ x_s \left(m_s \frac{d^2 y_s}{dt^2} - F_{sy} \right) - y_s \left(m_s \frac{d^2 x_s}{dt^2} - F_{sx} \right) \right\} = 0,$$

which is the same thing as

$$\frac{d}{dt} \sum_s m_s \left(x_s \frac{dy_s}{dt} - y_s \frac{dx_s}{dt} \right) = \sum_s (x_s F_{sy} - y_s F_{sx}),$$

(see § 6).

§ 8·2. Generalized Co-ordinates

We shall now introduce the generalized co-ordinates of § 6. The rectangular co-ordinates of any particle, s, of a system may be expressed in terms of the generalized co-ordinates, q, in the form

$$x_s = f_s (q_1, q_2, \ldots q_n),$$
$$y_s = g_s (q_1, q_2, \ldots q_n),$$
$$z_s = h_s (q_1, q_2, \ldots q_n),$$

in which f_s, g_s and h_s are given functions of the q's and the inherent constants of the dynamical system. We have in consequence

$$dx_s = \frac{\partial f_s}{\partial q_1} dq_1 + \frac{\partial f_s}{\partial q_2} dq_2 + \ldots + \frac{\partial f_s}{\partial q_n} dq_n,$$

and similar equations for dy_s and dz_s. It is convenient to use the symbol x_s itself to represent the functional dependence of the co-ordinate, x_s, on the q's. We therefore obtain

$$dx_s = \frac{\partial x_s}{\partial q_1} dq_1 + \frac{\partial x_s}{\partial q_2} dq_2 + \ldots + \frac{\partial x_s}{\partial q_n} dq_n,$$

$$dy_s = \frac{\partial y_s}{\partial q_1} dq_1 + \frac{\partial y_s}{\partial q_2} dq_2 + \ldots + \frac{\partial y_s}{\partial q_n} dq_n,$$

$$dz_s = \frac{\partial z_s}{\partial q_1} dq_1 + \frac{\partial z_s}{\partial q_2} dq_2 + \ldots + \frac{\partial z_s}{\partial q_n} dq_n. \qquad . \quad (8·2)$$

The symbol ' d ' will be used for increments which occur during the actual motion of the system, or during any motion we may tentatively ascribe to the system in the process of discovering the character of the actual motion. The symbol ' δ ' will be used for virtual displacements and the increments depending on them. The components, δx_1, δy_1, δz_1; δx_2, δy_2, δz_2; . . . of the virtual displacements of the particles of the system are not in general all arbitrary, as they may be subject to certain constraints. On the other hand the components, δq_1, δq_2, . . . δq_n, of a virtual displacement of the system are obviously quite

arbitrary, since the generalized co-ordinates are in fact so chosen as to be *independent* of one another. They thus satisfy the conditions imposed by the constraints, as it were, automatically.

If we replace the d's in (8·2) by δ's we get a corresponding set of equations for the virtual displacement (δx_s, δy_s, δz_s) of a particle in terms of the associated δq's.

The velocity of the particle, s, is given, in terms of the generalized velocities, by

$$\frac{dx_s}{dt} = \frac{\partial x_s}{\partial q_1}\dot{q}_1 + \frac{\partial x_s}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial x_s}{\partial q_n}\dot{q}_n \quad . \quad (8\cdot21)$$

where $\dot{q}$ means $\frac{dq}{dt}$. There are similar equations for $\frac{dy_s}{dt}$ and $\frac{dz_s}{dt}$. In these equations it will be observed that each differential quotient, $\frac{\partial x}{\partial q}$, $\frac{\partial y}{\partial q}$, . . . is expressed as a function of the q's and constants inherent in the system (§ 5·3).

By squaring $\frac{dx_s}{dt}$, $\frac{dy_s}{dt}$, $\frac{dz_s}{dt}$ respectively, adding and multiplying by m_s, the mass of the particle, we obtain twice its kinetic energy. Therefore if T represents the kinetic energy of the system

$$\begin{aligned} 2T = {}& Q_{11}\dot{q}_1\dot{q}_1 + Q_{12}\dot{q}_1\dot{q}_2 + \ldots + Q_{1n}\dot{q}_1\dot{q}_n \\ & + Q_{21}\dot{q}_2\dot{q}_1 + Q_{22}\dot{q}_2\dot{q}_2 + \ldots + Q_{2n}\dot{q}_2\dot{q}_n \\ & + \ldots . \\ & + Q_{n1}\dot{q}_n\dot{q}_1 + \ldots \qquad\qquad + Q_{nn}\dot{q}_n\dot{q}_n \quad . \quad (8\cdot22) \end{aligned}$$

in which Q_{12}, for example, means

$$Q_{12} = \sum_s m_s \left\{ \frac{\partial x_s}{\partial q_1}\frac{\partial x_s}{\partial q_2} + \frac{\partial y_s}{\partial q_1}\frac{\partial y_s}{\partial q_2} + \frac{\partial z_s}{\partial q_1}\frac{\partial z_s}{\partial q_2} \right\} \quad . \quad (8\cdot221)$$

Each Q is therefore expressed as a function of the q's and the inherent constants and it will be noticed that $Q_{\alpha\beta} = Q_{\beta\alpha}$.

It is convenient to abbreviate (8·22) by writing it in the form

$$2T = Q_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta \quad . \quad . \quad . \quad . \quad (8\cdot222)$$

in which the summation is indicated by the duplication or repetition of each of the subscripts α and β, and not by the symbol $\underset{\alpha \ \beta}{\Sigma\Sigma}$.

We see that (8·22) can be written in the form

$$2T = p_1\dot{q}_1 + p_2\dot{q}_2 + \ldots p_n\dot{q}_n \quad . \quad . \quad . \quad (8\cdot23)$$

or briefly $\qquad\qquad\qquad 2T = p_\alpha\dot{q}_\alpha$

where $\qquad\quad p_\alpha = Q_{\alpha1}\dot{q}_1 + Q_{\alpha2}\dot{q}_2 + \ldots + Q_{\alpha n}\dot{q}_n \quad . \quad . \quad (8\cdot24)$

or $\qquad\qquad\quad p_\alpha = Q_{\alpha\beta}\dot{q}_\beta$.

The quantity p_a is the **generalized momentum** corresponding to the co-ordinate q_a. Differentiating $2T$ partially with respect to $\dot{q}_a$, we get (see § 6·4),

$$p_a = \frac{\partial T}{\partial \dot{q}_a} \qquad \text{(8·241)}$$

If, for instance, we take $\dfrac{\partial T}{\partial \dot{q}_2}$, we might carry out the differentiation firstly along the second row of (8·22), thus obtaining

$$Q_{21}\dot{q}_1 + Q_{22}\dot{q}_2 + Q_{23}\dot{q}_3 + \ldots + Q_{2n}\dot{q}_n,$$

and then along the second vertical column, obtaining

$$Q_{12}\dot{q}_1 + Q_{22}\dot{q}_2 + Q_{32}\dot{q}_3 + \ldots Q_{n2}\dot{q}_n.$$

The two expressions are equal to one another (since $Q_{\alpha\beta} = Q_{\beta\alpha}$) and together make $2p_2$. Therefore

$$\frac{\partial(2T)}{\partial \dot{q}_2} = 2p_2,$$

in agreement with (8·241).

It is important to note that (8·22) expresses $2T$ *as a function of the q's, q̇'s and the inherent constants* of the system. It is a *quadratic* function of the $\dot{q}$'s. In (8·24) each p is also expressed as a function of the q's, $\dot{q}$'s and the inherent constants. It is a *linear* function of the $\dot{q}$'s.

From (8·24) we derive the equations

$$\dot{q}_1 = R_{11}p_1 + R_{12}p_2 + \ldots + R_{1n}p_n,$$
$$\dot{q}_2 = R_{21}p_1 + R_{22}p_2 + \ldots + R_{2n}p_n,$$
$$\cdots \cdots \cdots \cdots \cdots$$
$$\dot{q}_n = R_{n1}p_1 + R_{n2}p_2 + \ldots + R_{nn}p_n \qquad \text{(8·25)}$$

in which the $R_{\alpha\beta}$ are functions of the q's and inherent constants, and $R_{\alpha\beta} = R_{\beta\alpha}$. If we use the symbol $|Q|$ for the determinant

$$\begin{vmatrix} Q_{11}, & Q_{12}, & \ldots & Q_{1n} \\ Q_{21}, & Q_{22}, & \ldots & Q_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ Q_{n1}, & Q_{n2}, & \ldots & Q_{nn} \end{vmatrix},$$

and the symbol $|Q|_{\alpha\beta}$ for the determinant which is formed by omitting the row, α, and the column, β, each of these sub-determinants or minors having its sign so adjusted that, for example

$$|Q| = Q_{21}|Q|_{21} + Q_{22}|Q|_{22} + \ldots + Q_{2n}|Q|_{2n} \qquad \text{(8·251)}$$

then

$$R_{\alpha\beta} = \frac{|Q|_{\alpha\beta}}{|Q|}. \qquad \text{(8·252)}$$

Substituting the expressions (**8·25**) for $\dot{q}_a$ in (**8·23**), we get

$$2T = R_{11}p_1p_1 + R_{12}p_1p_2 + \ldots + R_{1n}p_1p_n$$
$$+ R_{21}p_2p_1 + R_{22}p_2p_2 + \ldots + R_{2n}p_2p_n$$
$$+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$+ R_{n1}p_np_1 + R_{n2}p_np_2 + \ldots + R_{nn}p_np_n \qquad (8·26)$$

Which expresses $2T$ as a function of the q's, the p's and the inherent constants.

From equation (**8·26**) we get

$$\frac{\partial T}{\partial p_a} = R_{a1}p_1 + R_{a2}p_2 + \ldots + R_{an}p_n$$

or
$$\frac{\partial T}{\partial p_a} = \dot{q}_a \quad . \quad . \quad . \quad . \quad (8·27)$$

by a process similar to that used to derive (**8·241**).

The work done, during a small displacement of a system, is

$$\underset{s}{\Sigma}(F_{sx}dx_s + F_{sy}dy_s + F_{sz}dz_s) ;$$

and therefore, when we substitute for the dx_s, dy_s, dz_s the expressions in (**8·2**), we get

$$\phi_1 dq_1 + \phi_2 dq_2 + \ldots + \phi_n dq_n,$$

or
$$\phi_a dq_a . \quad . \quad . \quad . \quad . \quad . \quad (8·28)$$

in which it is easily seen that

$$\phi_a = \underset{s}{\Sigma}\left(F_{sx}\frac{\partial x_s}{\partial q_a} + F_{sy}\frac{\partial y_s}{\partial q_a} + F_{sz}\frac{\partial z_s}{\partial q_a}\right) \quad . \qquad (8·281)$$

We may term ϕ_1, ϕ_2, etc. the **generalized forces** corresponding to the co-ordinates q_1, q_2, etc.

§ 8·3. PRINCIPLE OF ENERGY

The use of the *term* 'energy' is of comparatively recent origin ; but the *conception* of energy began to emerge as far back as the time of Huygens (1629–1695). In § 5·1 we deduced from Newton's laws that the increase of the kinetic energy, $\frac{1}{2}mv^2$, of a particle is equal to the work done by the force or forces acting on it (**5·12**). In any mechanical system whatever the work done by the forces of the system is equal to the corresponding increase of its kinetic energy. This is the principle of energy in the form (principle of vis viva) which is of peculiar importance in mechanics, i.e. in connexion with problems in which we are concerned only with movements of material masses under the influence of forces, explicitly given, or due to constraints or analogous causes.

The further development of the conception of energy is linked up with discoveries in different directions. In many mechanical problems, some of which we have already dealt with, the work done by the forces can be equated to the decrement of a certain quantity, V (§ 5·1), a function of the co-ordinates of the system, or a function, we may say, of its configuration. Since this is equal to the increment of the kinetic energy, T, a function of the state of motion of the system and its configuration, the sum $T + V$ remains unaltered. The work is done at the expense of V and results in an equal increase of T. Then the consistent failure of all attempts to devise a machine (perpetuum mobile) capable of doing work gratis, and the success, on the other hand, in devising machines capable of doing work by the consumption of coal, gas or oil, gradually produced the conviction that work can be done only at *some expense*; that whenever work is done, *something* is necessarily consumed. This *something* is called **energy,** and we conventionally adopt the amount of work done as a measure of the energy consumed. This does not mean that the energy of a body or a system is merely its *capacity for doing work*. There is some reason to believe that energy has a more substantial character, more *perseity* than is suggested by ' capacity for doing work '.

Finally the experimental work of a long line of investigators, Count Rumford, Davy, Colding, Hirn and above all, Joule, established that when **heat** [1] is generated by doing work, as for example in overcoming friction, and alternatively when work is done, as in the case of the steam engine, at the expense of heat, the quantity of heat (generated or consumed as the case may be) is proportional to the work done; the factor of proportionality (mechanical equivalent of heat) being the *same*, whether work is done at the expense of heat or heat produced in consequence of work done. This suggested that the heat in a body should be identified with the kinetic energy (or kinetic and potential energy) of the particles (molecules) of which it is constituted, and gave rise to the modern **Principle of Conservation of Energy,** according to which the energy in the world remains invariable in quantity. The constancy of $T + V$ in certain mechanical systems is merely a special case therefore of the wider energy principle, and in the middle period of last century, and still later, it was generally held that, not only heat, but all other forms of energy were either kinetic energy or potential energy in the sense in which these terms are used in mechanics.

The principle of conservation of energy is in excellent accord with the view, which until quite recent times was universally

[1] Heat measured by the use of mercury thermometers. See § 15·5.

held, that physical and chemical phenomena are *au fond*
mechanical phenomena ; and almost till the closing years of the
century physical theories were held to be satisfactory or other-
wise, just in proportion to the degree of success with which they
furnished a mechanical picture of the Newtonian type.

§ 8.4.   Equations of Hamilton and Lagrange

If a function
$$V = V(q_1, q_2, \ldots q_n) . \quad \ldots \quad \text{(8·4)}$$
exist, such that
$$\phi_1 = -\frac{\partial V}{\partial q_1},$$
$$\phi_2 = -\frac{\partial V}{\partial q_2}, \quad \ldots \quad \text{(8·401)}$$
$$\cdot \quad \cdot \quad \cdot$$
$$\phi_n = -\frac{\partial V}{\partial q_n},$$

where $\phi_1, \phi_2, \ldots \phi_n$ are the generalized forces (**8·28**), the work
done by them, during a small displacement of the system, will be

$$-\frac{\partial V}{\partial q_1}dq_1 - \frac{\partial V}{\partial q_2}dq_2 \ldots - \frac{\partial V}{\partial q_n}dq_n \quad \text{(8·402)}$$

or
$$-\frac{\partial V}{\partial q_a}dq_a.$$

This must be equal to the corresponding increase of the kinetic
energy, T.   Therefore

$$dT = -\frac{\partial V}{\partial q_a}dq_a \quad \ldots \quad \text{(8·41)}$$

and in consequence of (8·4)
$$dT = -dV$$
or
$$d(T + V) = 0 ;$$

so that the mechanical energy, $T + V$, remains constant.   Such
a system is said to be **conservative**.   This is the exceptional
case.   In general $T + V$ varies.   This may happen in conse-
quence of a complementary variation of the $T + V$ of some other
system, or it may be associated with the development of heat,
as when there are frictional forces, or with variations of other
forms of energy.

Instead of confining our attention to conservative systems,
let us suppose that there is a potential energy function, V, such
that the generalized forces are given by (**8·401**) ; but that V
has the form

$$V = V(q_1, q_2, \ldots q_n \; t) \quad \ldots \quad \text{(8·42)}$$

Equation (**8·41**) will still hold, but since

$$dV = \frac{\partial V}{\partial q_\alpha}dq_\alpha + \frac{\partial V}{\partial t}dt,$$

we have

$$d(T + V) = \frac{\partial V}{\partial t}dt,$$

or

$$\frac{d(T + V)}{dt} = \frac{\partial V}{\partial t} \quad . \quad . \quad . \quad . \quad (\mathbf{8\cdot421})$$

As the time does not appear explicitly in the expression for T, whether we take (**8·22**) or (**8·26**), we get from (**8·421**)

$$\frac{d(T + V)}{dt} = \frac{\partial(T + V)}{\partial t} \quad . \quad . \quad . \quad (\mathbf{8\cdot422})$$

When T is expressed as a function of the generalized momenta (**8·26**) we shall represent $T + V$ by the symbol H, so that H does not merely denote the energy, $T + V$, but it is also a *functional symbol*. Since V does not contain the p's it is clear that

$$\frac{\partial H}{\partial p_\alpha} = \frac{\partial T}{\partial p_\alpha} \quad . \quad . \quad . \quad . \quad (\mathbf{8\cdot423})$$

and therefore by (**8·27**)

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$$

or

$$\boxed{\frac{dq_\alpha}{dt} = \frac{\partial H}{\partial p_\alpha}} \quad . \quad . \quad . \quad . \quad (\mathbf{8\cdot43})$$

It is essential that we should bear in mind that the partial differential quotient $\dfrac{\partial T}{\partial q_\alpha}$, *T being expressed as a function of the p's and q's*, is quite different from $\dfrac{\partial T}{\partial q_\alpha}$ obtained from *T expressed as a function of the $\dot{q}$'s and q's*. In fact, the former differentiation is subjected to the condition that the p's and the remainder of the q's are kept constant, while the latter is subjected to the condition that the $\dot{q}$'s and the remainder of the q's are kept constant. To avoid confusion let us write

$$\frac{\partial T(p, q)}{\partial q} = \frac{DT}{Dq},$$

and

$$\frac{\partial T(\dot{q}, q)}{\partial q} = \frac{\partial T}{\partial q}.$$

8

We may express a small change δT in the kinetic energy of a system in the following different ways :—

$$2\delta T = p_a \delta \dot{q}_a + \dot{q}_a \delta p_a \quad . \quad . \quad . \quad . \quad \text{(by 8·23)}$$

$$\delta T = \frac{\partial T}{\partial \dot{q}_a} \delta \dot{q}_a + \frac{\partial T}{\partial q_a} \delta q_a,$$

$$\delta T = \frac{DT}{Dp_a} \delta p_a + \frac{DT}{Dq_a} \delta q_a \quad . \quad . \quad . \quad . \quad \text{(8·44)}$$

Subtracting the last of these from the first, and replacing $\dfrac{DT}{Dp_a}$ by $\dot{q}_a$ (8·27), we get

$$\delta T = p_a \delta \dot{q}_a - \frac{DT}{Dq_a} \delta q_a, \quad . \quad . \quad . \quad \text{(8·441)}$$

and the second equation (8·44) may be expressed in the form

$$\delta T = p_a \delta \dot{q}_a + \frac{\partial T}{\partial q_a} \delta q_a \quad . \quad . \quad . \quad \text{(8·442)}$$

since $p_a = \dfrac{\partial T}{\partial \dot{q}_a}$. Hence by comparison of (8·441) and (8·442) we find

$$\frac{DT}{Dq_a} = -\frac{\partial T}{\partial q_a} \quad . \quad . \quad . \quad . \quad . \quad \text{(8·45)}$$

Adding $\dfrac{\partial V}{\partial q_a}$ to both sides of this equation, we obtain

$$\frac{D(T + V)}{Dq_a} = -\frac{\partial(T - V)}{\partial q_a}$$

which we may put in the form

$$\frac{\partial H}{\partial q_a} = -\frac{\partial L}{\partial q_a}. \quad . \quad . \quad . \quad \text{(8·451)}$$

where $L = T - V$ is also a functional symbol indicating that T is expressed as a function of the q's and $\dot{q}$'s. On the left of this equation D/Dq_a has been replaced by $\partial/\partial q_a$, since the functional symbol H already indicates that the T in it is a function of the q's and p's.

From (8·41) we have

$$\frac{dT}{dt} + \frac{\partial V}{\partial q_a}\frac{dq_a}{dt} = 0,$$

therefore

$$\frac{\partial T}{\partial p_a}\frac{dp_a}{dt} + \frac{D(T + V)}{Dq_a}\frac{dq_a}{dt} = 0,$$

and replacing $\dfrac{\partial T}{\partial p_a}$ by $\dfrac{dq_a}{dt}$ (8·27), we obtain

$$\left(\frac{dp_a}{dt} + \frac{\partial H}{\partial q_a}\right)\frac{dq_a}{dt} = 0,$$

and consequently also

$$\left(\frac{dp_a}{dt} - \frac{\partial L}{\partial q_a}\right)\frac{dq_a}{dt} = 0.$$

(See **8·451**).

This suggests, though it does not prove, the equations

$$\boxed{\frac{dp_a}{dt} + \frac{\partial H}{\partial q_a} = 0} \qquad . \quad . \quad . \quad . \quad \textbf{(8·46)}$$

and

$$\frac{dp_a}{dt} - \frac{\partial L}{\partial q_a} = 0 \qquad . \quad . \quad . \quad . \quad \textbf{(8·461)}$$

Their validity will be established in § 8·6. The equations **8·46** together with (**8·43**) are known as **Hamilton's canonical equations.** The equations (**8·461**) are the **equations of Lagrange** and are usually written in the form

$$\boxed{\frac{d}{dt}\!\left(\frac{\partial L}{\partial \dot{q}_a}\right) - \frac{\partial L}{\partial q_a} = 0} \qquad . \quad . \quad . \quad \textbf{(8·462)}$$

We may write them in this way, because V does not contain the $\dot{q}$'s and therefore

$$p_a = \frac{\partial T}{\partial \dot{q}_a} = \frac{\partial (T - V)}{\partial \dot{q}_a} = \frac{\partial L}{\partial \dot{q}_a}.$$

The function L is called the **Lagrangian Function.**

§ 8·5. Illustrations. Cyclic Co-ordinates

As a first illustration we may take the case of the compound pendulum § 6·5. Here we have one q, which is, conveniently, the angle θ, Fig. 6·5.

The energy equation is (**6·501**)

$$\tfrac{1}{2}I\!\left(\frac{d\theta}{dt}\right)^{2} - Mgh \cos\theta = E \qquad . \quad . \quad \textbf{(8·5)}$$

and system is conservative. The corresponding p is

$$p = \frac{\partial T}{\partial \dot{\theta}} = I\frac{d\theta}{dt} \qquad . \quad . \quad . \quad . \quad \textbf{(8·501)}$$

Therefore
$$H = \frac{p^2}{2I} - Mgh \cos \theta$$

and the canonical equations are

$$\frac{dp}{dt} = - \frac{\partial}{\partial \theta}\left(\frac{p^2}{2I} - Mgh \cos \theta\right),$$

$$\frac{d\theta}{dt} = \frac{\partial}{\partial p}\left(\frac{p^2}{2I} - Mgh \cos \theta\right);$$

whence we obtain

$$\frac{d\theta}{dt} = \frac{p}{I}$$

and therefore

$$I\frac{d^2\theta}{dt^2} = - Mgh \sin \theta,$$

in agreement with (6·5).

The Lagrangian function for the pendulum is

$$L = \tfrac{1}{2}I\dot{\theta}^2 + Mgh \cos \theta,$$

and consequently the Lagrangian equation is

$$\frac{d}{dt} \frac{\partial}{\partial \dot{\theta}} \{\tfrac{1}{2}I\dot{\theta}^2 + Mgh \cos \theta\} - \frac{\partial}{\partial \theta} \{\tfrac{1}{2}I\dot{\theta}^2 + Mgh \cos \theta\} = 0,$$

whence
$$I\frac{d^2\theta}{dt^2} + Mgh \sin \theta = 0$$

as before.

It will be noticed that when there are n degrees of freedom there are $2n$ canonical equations of the *first* order while there are n Lagrangian equations of the *second* order.

The case of the pendulum is merely illustrative. It is clear that nothing is gained by the equations of Hamilton or Lagrange in cases like this. Having set up the energy equation, it can only be described as a retrograde step to differentiate it. It is when we come to systems with *more* than one degree of freedom that the merits of the methods of Hamilton and Lagrange begin to appear.

Let us turn to the case, § 5·5, of a particle moving under the influence of a central force $\dfrac{B}{r^2}$. We get (see **5·51**) for the Hamiltonian function

$$H = \frac{p_r{}^2}{2m} + \frac{p_\theta{}^2}{2mr^2} + \frac{B}{r} \quad . \quad . \quad . \quad . \quad (8\cdot51)$$

one of the q's is the radial distance, r, and the other is the angle, θ.

In this case Hamilton's equations become

$$\frac{dp_r}{dt} = -\frac{\partial}{\partial r}\left(\frac{p_r{}^2}{2m} + \frac{p_\theta{}^2}{2mr^2} + \frac{B}{r}\right),$$

$$\frac{dp_\theta}{dt} = -\frac{\partial}{\partial \theta}\left(\frac{p_r{}^2}{2m} + \frac{p_\theta{}^2}{2mr^2} + \frac{B}{r}\right),$$

$$\frac{dr}{dt} = \frac{\partial}{\partial p_r}\left(\frac{p_r{}^2}{2m} + \frac{p_\theta{}^2}{2mr^2} + \frac{B}{r}\right),$$

$$\frac{d\theta}{dt} = \frac{\partial}{\partial p_\theta}\left(\frac{p_r{}^2}{2m} + \frac{p_\theta{}^2}{2mr^2} + \frac{B}{r}\right).$$

On carrying out the partial differentiations we get

$$\frac{dp_r}{dt} = \frac{p_\theta{}^2}{mr^3} + \frac{B}{r^2},$$

$$\frac{dp_\theta}{dt} = 0,$$

$$\frac{dr}{dt} = \frac{p_r}{m},$$

$$\frac{d\theta}{dt} = \frac{p_\theta}{mr^2}.$$

From these four first order equations we may derive the following two second order equations :

$$m\frac{d^2r}{dt^2} = \frac{p_\theta{}^2}{mr^3} + \frac{B}{r^2},$$

$$\frac{d}{dt}\left\{mr^2\frac{d\theta}{dt}\right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8\cdot52)$$

This example illustrates two points : (i) The two equations we have obtained are sufficient, since the object may be said to be to express r and θ as functions of the time. We have already one equation, the energy equation, at the very outset, and therefore we do not need both the equations (8·52) which we have derived. Instead of employing for the final solution of the dynamical problem the equations (8·52), it is preferable to use the energy equation and one of them. The reason for this is that the energy equation has already advanced one step in the series of integrations marking the way to the final goal, the accompanying constant of integration being in fact the most important of all, namely the energy. (ii) Whenever one or more of the co-ordinates do not appear explicitly in the function H, as for example θ in the problem of the motion of a particle under a central force, the corresponding momentum is constant. Such co-ordinates are termed **cyclic co-ordinates.**

The Lagrangian function derived from (5·51) is

$$L = \tfrac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \tfrac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2 - \frac{B}{r},$$

or

$$L = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - \frac{B}{r}.$$

Therefore

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r},$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta},$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + \frac{B}{r^2},$$

$$\frac{\partial L}{\partial \theta} = 0,$$

and consequently

$$m\frac{d^2r}{dt^2} = mr\dot{\theta}^2 + \frac{B}{r^2},$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0,$$

in agreement with (8·52).

For another illustration we may turn to the problem of the spinning top. The energy equation (see § 7·3) gives us

$$\tfrac{1}{2}A\{\sin^2\theta.\dot{\psi}^2 + \dot{\theta}^2\}$$
$$+ \tfrac{1}{2}C\{\dot{\phi} + \cos\theta\dot{\psi}\}^2 + Mgh\cos\theta = E \quad . \quad (8·53)$$

from which we find

$$p_\psi = A\sin^2\theta.\dot{\psi} + C(\dot{\phi} + \cos\theta.\dot{\psi})\cos\theta,$$
$$p_\theta = A\dot{\theta},$$
$$p_\phi = C\{\dot{\phi} + \cos\theta.\dot{\psi}\}.$$

Hence

$$H = \frac{(p_\psi - \cos\theta p_\phi)^2}{2A\sin^2\theta} + \frac{p_\theta^2}{2A} + \frac{p_\phi^2}{2C} + Mgh\cos\theta \quad . \quad (8·531)$$

and

$$L = \tfrac{1}{2}A(\sin^2\theta.\dot{\psi}^2 + \dot{\theta}^2) + \tfrac{1}{2}C(\dot{\phi} + \cos\theta.\dot{\psi})^2 - Mgh\cos\theta \quad (8·532)$$

Whether we employ the equations of Hamilton or those of Lagrange, we find

$$p_\psi = \text{constant},$$
$$p_\phi = \text{constant},$$

and these, together with the energy equation (8·53) are equivalent to (7·31), (7·32) and (7·33) which we have found already.

The preceding examples illustrate conservative systems, in

which the potential energy, V, does not contain the time. The following example furnishes a simple illustration of a non-conservative Hamiltonian system. A particle of mass m is constrained to keep to a straight line, and subject to a restoring force proportional to its displacement from a fixed point, O, plus a force which is a simple harmonic function of the time. Its equation of motion will be

$$m\frac{d^2q}{dt^2} = -\mu q + R\cos\omega t$$

where μ, R and ω are constants. In this case

$$T = \tfrac{1}{2}m\dot{q}^2 = \frac{p^2}{2m}$$

and

$$V = \frac{\mu}{2}q^2 - qR\cos\omega t$$

since V is defined to fulfil the condition

$$\text{force} = -\frac{\partial V}{\partial q}.$$

The Hamiltonian function is therefore

$$H = \frac{p^2}{2m} + \frac{\mu}{2}q^2 - qR\cos\omega t,$$

and the Lagrangian function,

$$L = \tfrac{1}{2}m\dot{q}^2 - \frac{\mu}{2}q^2 + qR\cos\omega t.$$

§ 8·6. Principles of Action

If we have to deal with a system of not more than two degrees of freedom we may repre-sent its configuration and be-haviour graphically, by rectan-gular axes of co-ordinates using lengths measured from the origin along two of the axes to repre-sent the values of the q's and a length measured along the re-maining axis to represent the corresponding time (Fig. 8·6). The motion of the system will be completely represented by a

Fig. 8·6

line such as (1, 2) in the diagram. We shall use the methods and the language which are appropriate for this graphical repre-sentation for systems of any number of degrees of freedom.

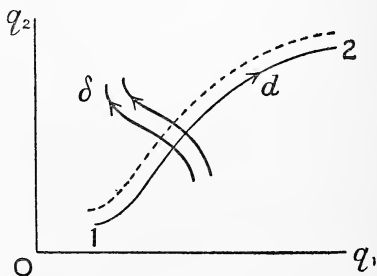The principle of d'Alembert (**8·11**), if applied to the type

of dynamical system dealt with in §§ 8·4 and 8·5, will take the form

$$m_s\left(\frac{d^2x_s}{dt^2}\delta x_s + \frac{d^2y_s}{dt^2}\delta y_s + \frac{d^2z_s}{dt^2}\delta z_s\right) + \frac{\partial V}{\partial q_a}\delta q_a = 0 \quad . \quad . \quad (8\cdot6)$$

In this equation the summations are sufficiently indicated by the repetitions of s and a. The s summation extends over all the particles of the system. The δx_s, δy_s, . . . δq_a . . . represent virtual displacements. Our purpose is to investigate the actual motion of the system (represented by the path (1, 2) in Fig. 8·6) by studying its relation to motions represented by neighbouring paths (such as that shown in the figure by a broken line). These lines are comparable with the **d** lines of Fig. 3·31 in the proof of the theorem of Stokes, and we may conveniently suppose them to be drawn on a surface. It is helpful to regard the virtual displacements, δq_a, as given by the intersections of this surface by a family of surfaces,

$$f(q_1, q_2, . . . q_n, t) = C \quad . \quad . \quad . \quad . \quad (8\cdot61)$$

These are quite arbitrarily chosen surfaces on account of the arbitrariness of the virtual displacements, δq_a. If we pick out one of them by giving the constant C any value λ, a neighbouring surface will be one for which

$$C = \lambda + d\lambda.$$

The lines of intersection of this family of surfaces with the surface on which the **d** lines lie we shall naturally call **δ** lines, as in § 3·3. The symbol δ will therefore represent an increment incurred in passing along a **δ** line from the **d** line representing the actual motion to the neighbouring **d** line ; while the symbol d will represent an increment incurred in passing along a **d** line from a surface $C = \lambda$ to a neighbouring surface $C = \lambda + d\lambda$, i.e. from one **δ** line to the next.

We shall now make use of a device, already employed in previous investigations, namely that embodied in the formula

$$a\frac{db}{dt} = \frac{d(ab)}{dt} - b\frac{da}{dt}.$$

Substituting for a and b respectively,

$$\delta x_s \text{ and } m_s\frac{dx_s}{dt},$$

we get

$$m_s\frac{d^2x_s}{dt^2}\delta x_s = \frac{d}{dt}\left\{m_s\frac{dx_s}{dt}\delta x_s\right\} - m_s\frac{dx_s}{dt}\frac{d}{dt}(\delta x_s), \quad . \quad (8\cdot611)$$

and similar expressions for

$$m_s\frac{d^2y_s}{dt^2}\delta y_s \text{ and } m_s\frac{d^2z_s}{dt^2}\delta z_s.$$

Equation (8·6) thus becomes

$$\frac{d}{dt}\left\{ m_s\frac{dx_s}{dt}\delta x_s + m_s\frac{dy_s}{dt}\delta y_s + m_s\frac{dz_s}{dt}\delta z_s \right\}$$

$$- \left\{ m_s\frac{dx_s}{dt}\frac{d}{dt}(\delta x_s) + m_s\frac{dy_s}{dt}\frac{d}{dt}(\delta y_s) + m_s\frac{dz_s}{dt}\frac{d}{dt}(\delta z_s) \right\}$$

$$+ \frac{\partial V}{\partial q_\alpha}\delta q_\alpha = 0,$$

or, (§ 8·2),

$$\frac{d}{dt}\{p_\alpha\delta q_\alpha\} - \left\{ m_s\frac{dx_s}{dt}\frac{d}{dt}(\delta x_s) + m_s\frac{dy_s}{dt}\frac{d}{dt}(\delta y_s) + m_s\frac{dz_s}{dt}\frac{d}{dt}(\delta z_s) \right\}$$

$$+ \frac{\partial V}{\partial q_\alpha}\delta q_\alpha = 0 \qquad (8·612)$$

We have seen (3·314) that $d\delta x = \delta dx$, etc., and we may show in a similar way that $d\delta t = \delta dt$; but *it does not follow*, for example, that $\frac{d}{dt}(\delta x) = \delta\left(\frac{dx}{dt}\right)$, We have in fact (Fig. 8·61)

$$\delta\left(\frac{dx}{dt}\right) = \frac{d(x + \delta x)}{d(t + \delta t)} - \frac{dx}{dt},$$

$$\delta\left(\frac{dx}{dt}\right) = \frac{dt\,d\delta x - dx\,d\delta t}{dt\,dt + dt\,d\delta t},$$

or $\delta\left(\frac{dx}{dt}\right) = \frac{d\delta x}{dt} - \frac{dx}{dt}\frac{d\delta t}{dt}$. (8·62)

and similar formulæ for

$$\delta\left(\frac{dy}{dt}\right) \text{ and } \delta\left(\frac{dz}{dt}\right).$$
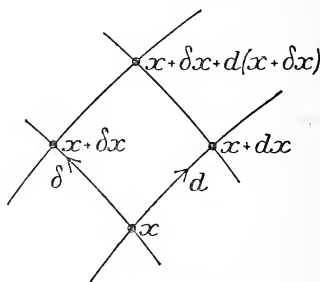
FIG. 8·61

With the aid of (8·62) we may now express (8·612) in the form

$$\frac{d}{dt}\{p_\alpha\delta q_\alpha\} - \left\{ m_s\frac{dx_s}{dt}\delta\left(\frac{dx_s}{dt}\right) + m_s\frac{dy_s}{dt}\delta\left(\frac{dy_s}{dt}\right) + m_s\frac{dz_s}{dt}\delta\left(\frac{dz_s}{dt}\right) \right\}$$

$$- \left\{ m_s\left(\frac{dx_s}{dt}\right)^2 + m_s\left(\frac{dy_s}{dt}\right)^2 + m_s\left(\frac{dz_s}{dt}\right)^2 \right\}\frac{d\delta t}{dt} + \frac{\partial V}{\partial q_\alpha}\delta q_\alpha = 0,$$

or

$$\frac{d}{dt}\{p_\alpha\delta q_\alpha\} - \delta T - 2T\frac{d\delta t}{dt} + \frac{\partial V}{\partial q_\alpha}\delta q_\alpha = 0,$$

and therefore (§ 8·4)

$$\frac{d}{dt}\{p_\alpha\delta q_\alpha\} - \delta T - 2T\frac{d\delta t}{dt} + \delta V - \frac{\partial V}{\partial t}\delta t = 0,$$

and by (8·421)

$$\frac{d}{dt}\{p_\alpha\delta q_\alpha\} - \delta T - 2T\frac{d\delta t}{dt} + \delta V - \frac{dE}{dt}\delta t = 0.$$

Now
$$\frac{d}{dt}(E\delta t) = \frac{dE}{dt}.\delta t + E\frac{d\delta t}{dt};$$

therefore we find
$$\frac{d}{dt}\{p_a\delta q_a - E\delta t\} - \delta T - 2T\frac{d\delta t}{dt} + \delta V + E\frac{d\delta t}{dt} = 0,$$

or
$$\frac{d}{dt}\{p_a\delta q_a - E\delta t\} - \delta(2T - E) - (2T - E)\frac{d\delta t}{dt} = 0.$$

If we multiply this equation by dt and integrate between the limits 1 and 2 we obtain

$$\left| p_a\delta q_a - E\delta t \right|_1^2 - \int_1^2 \{dt\delta(2T - E) + (2T - E)\delta dt\} = 0,$$

or finally

$$\left| p_a\delta q_a - E\delta t \right|_1^2 - \delta\int_1^2 (2T - E)dt = 0 \quad . \quad . \quad (8 \cdot 63)$$

In this equation the variations symbolized by δ are subject to no conditions, except that they are small.

Let us give our attention in the first place to conservative systems, i.e. systems for which $dE = 0$. Since the variations in $(8 \cdot 63)$ are arbitrary we may subject them to the condition $\delta E = $ a constant. We then have

$$\left| p_a\delta q_a - E\delta t \right|_1^2 - \delta\int_1^2 2Tdt + \delta E\int_1^2 dt + E\int_1^2 d\delta t = 0,$$

or
$$\left| p_a\delta q_a \right|_1^2 + \delta E(t_2 - t_1) = \delta\int_1^2 2Tdt \quad . \quad . \quad (8 \cdot 631)$$

If we suppose the two paths to join at the lower limit 1 but not at the upper limit 2, we get, on dropping the index 2,

$$p_a\delta q_a + (t - t_1)\delta E = \delta\int_1 2Tdt. \quad . \quad . \quad (8 \cdot 632)$$

or, if we use the symbol A for the integral on the right,
$$p_a\delta q_a + (t - t_1)\delta E = \delta A \quad . \quad . \quad . \quad (8 \cdot 633)$$

The function A is one of those to which the term **action** is applied and $(8 \cdot 633)$ indicates that it may be expressed as a function of the q's and E, and therefore

$$p_a = \frac{\partial A}{\partial q_a},$$

$$t - t_1 = \frac{\partial A}{\partial E} \quad . \quad . \quad . \quad . \quad . \quad (8 \cdot 634)$$

If the system is strictly periodic and the range of integration,

$\int_1^2$, extends exactly over the period, τ, of the system,

$$\tau = t_2 - t_1,$$

the terms $\mid p_a \delta q_a \overset{2}{\underset{1}{\mid}}$ must vanish, and we find

$$(t_2 - t_1)\delta E = \delta A,$$

or if we denote this particular value of A by the letter J,

$$\left.\begin{array}{l}\tau\delta E = \delta J \\ \delta E = \nu\delta J\end{array}\right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \textbf{(8·635)}$$

or

In the next place let us suppose the two paths to be co-terminous in space (not necessarily in time) so that $\mid p_a \delta q_a \overset{2}{\underset{1}{\mid}} = 0$, since the terminal δq's vanish. Then if the variations are subjected to the condition $\delta E = 0$ we find

$$\delta \int_1^2 2T dt = 0 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \textbf{(8·636)}$$

for systems for which $dE = 0$, i.e. for conservative systems. This is the **principle of least action** in its original form. It was first given in 1747 by de Maupertuis, a Frenchman, who was, for a time, president of the Royal Prussian Academy during the reign of Frederick the Great. He claimed for his principle a foundation in the attributes of the Deity. ' Notre principe . . . est une suite nécessaire de l'emploi le plus sage de cette puissance,' i.e. ' la puissance du Createur,' and the principle has turned out to be not unworthy of the claim made for it. A better name for it would be ' principle of *stationary* action ', since the action $\int_1^2 2T dt$ is not in all cases a *minimum*.

If in **(8·63)** we suppose the two paths to be co-terminous in space *and time*, i.e. the terminal variations δq_a and δt are all zero, we get

$$\delta \int_1^2 (2T - E)dt = 0$$

or

$$\delta \int_1^2 (\text{T} - V)dt = 0 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \textbf{(8·64)}$$

This form (the most important one) of the principle of action is known as **Hamilton's principle.** The function

$$S = \int (T - V)dt \quad . \quad . \quad . \quad . \quad . \quad \textbf{(8·65)}$$

is called **Hamilton's principal function,** while the function A (**8·633**) is called **Hamilton's characteristic function.**

If we take the two paths to be co-terminous (in space and time) at the lower limit only we get from (**8·63**), dropping the upper index, 2,

$$p_a \delta q_a - E \delta t = \delta S \quad . \quad . \quad . \quad . \quad \textbf{(8·66)}$$

and therefore S is a function of the q's and the time and

$$p_a = \frac{\partial S}{\partial q_a}, \quad - E = \frac{\partial S}{\partial t} \quad . \quad . \quad . \quad \textbf{(8·661)}$$

We may use Hamilton's principle (**8·64**) to establish the canonical equations and the equations of Lagrange. If we express E as a function of the p's, q's and t it becomes

$$\delta \int_1^2 (2T - H)dt = 0$$

or

$$\delta \int_1^2 (p_a \dot{q}_a - H)dt = 0.$$

Since the variations δ are perfectly arbitrary, it is permissible to subject them to the condition $\delta t = 0$. With this condition (**8·62**) becomes

$$\delta\left(\frac{dq}{dt}\right) = \frac{d\delta q}{dt}$$

or

$$\delta \dot{q} = \frac{d}{dt}\delta q \quad . \quad . \quad . \quad . \quad \textbf{(8·662)}$$

We therefore find

$$\int_1^2 \left(p_a \delta \dot{q}_a + \dot{q}_a \delta p_a - \frac{\partial H}{\partial p_a}\delta p_a - \frac{\partial H}{\partial q_a}\delta q_a\right)dt = 0 \quad . \quad . \quad \textbf{(8·67)}$$

But we have proved, (**8·43**), that

$$\dot{q}_a = \frac{\partial H}{\partial p_a},$$

therefore (**8·67**) becomes

$$\int_1^2 \left(p_a \delta \dot{q}_a - \frac{\partial H}{\partial q_a}\delta q_a\right)dt = 0,$$

or
$$\int_1^2 \left(p_a \frac{d}{dt}\delta q_a - \frac{\partial H}{\partial q_a}\delta q_a \right) dt = 0,$$

by **(8·662)**.

Therefore
$$\int_1^2 \left(\frac{d}{dt}(p_a \delta q_a) - \delta q_a \frac{dp_a}{dt} - \frac{\partial H}{\partial q_a}\delta q_a \right) dt = 0.$$

Now the integral
$$\int_1^2 \frac{d}{dt}(p_a \delta q_a)dt = \left| p_a \delta q_a \right|_1^2 = 0,$$

since the paths intersect at 1 and 2, therefore
$$\int_1^2 \delta q_a \left(\frac{dp_a}{dt} + \frac{\partial H}{\partial q_a} \right) dt = 0.$$

As the δq_a are arbitrary this result requires that
$$\frac{dp_a}{dt} + \frac{\partial H}{\partial q_a} = 0.$$

These are the canonical equations of Hamilton. Those of Lagrange follow immediately, since
$$\frac{\partial H}{\partial q_a} = -\frac{\partial L}{\partial q_a}.$$

There is a certain function $H(p_a, q_a, t)$ which is equal to $T + V$ or to E, i.e.
$$H(p_a, q_a, t) - E = 0 \quad . \quad . \quad . \quad . \text{ (8·675)}$$
and if we substitute for E and the p's the expressions in **(8·661)** we get **Hamilton's partial differential equation**
$$\boxed{H\left(\frac{\partial S}{\partial q_a}, q_a, t\right) + \frac{\partial S}{\partial t} = 0} \quad . \quad . \quad . \quad . \text{ (8·68)}$$

When E is constant $= a$ say, the equation becomes
$$H\left(\frac{\partial S}{\partial q_a}, q_a\right) = a,$$

as H does not contain the time explicitly; or, since
$$\frac{\partial S}{\partial q_a} = \frac{\partial A}{\partial q_a},$$
$$H\left(\frac{\partial A}{\partial q_a}, q_a\right) = a \quad . \quad . \quad . \quad . \text{ (8·681)}$$

§ 8·7. Jacobi's Theorem.

Hamilton's principal function, S, defined by (8·65), is a function of the co-ordinates, q, and the time ; and it satisfies the partial differential equation (8·68). A converse proposition naturally suggests itself. Having set up the energy equation (8·675), appropriate to a dynamical problem, and derived the partial differential equation by replacing each p_a by the corresponding $\dfrac{\partial S}{\partial q_a}$ and E by $-\dfrac{\partial S}{\partial t}$; let us suppose an integral S to have been found. Will the differential quotients, $\pi_a = \dfrac{\partial S}{\partial q_a}$, be identical with the corresponding generalized momenta of the dynamical system ? This amounts to asking if it is a matter of indifference whether we use for S a solution of (8·68) or the function (8·65).

We shall prove that this is the case *provided we use a solution which is a* **complete integral** *of* (8·68). This is an integral containing as many arbitrary constants as there are independent variables. It must be distinguished from a **singular integral,** which is a relation between the variables *involving no arbitrary constant,* and moreover is not a particular case of the complete integral ; and from a **general integral** which involves arbitrary *functions* and therefore altogether transcends the other integrals in its generality.

The complete integral of (8·68) will have $n + 1$ arbitrary constants, if we suppose there are n co-ordinates, q. We shall represent them by a_1, a_2, $a_3 \ldots a_n$, a_{n+1}. One of them, which we shall take to be a_{n+1}, is merely additive. If S be a complete integral, we shall prove that the equations

$$\frac{\partial S}{\partial a_1} = \beta_1, \qquad \frac{\partial S}{\partial q_1} = \pi_1,$$

$$\frac{\partial S}{\partial a_2} = \beta_2, \qquad \frac{\partial S}{\partial q_2} = \pi_2,$$

$$\cdots \quad (A) \quad \cdots \quad (B)$$

$$\frac{\partial S}{\partial a_n} = \beta_n, \qquad \frac{\partial S}{\partial q_n} = \pi_n \quad \cdots \quad (8·7)$$

constitute a solution of the associated dynamical problem, if β_1, β_2, $\ldots \beta_n$ are arbitrary constants, and if we identify π_1, π_2, $\ldots \pi_n$ with the generalized momenta, p_1, p_2, $\ldots p_n$

respectively. Since $\beta_1, \beta_2, \ldots \beta_n$ are constants, we have from the equations A (8·7)

$$\frac{d}{dt}\left(\frac{\partial S}{\partial a_1}\right) = 0,$$

$$\frac{d}{dt}\left(\frac{\partial S}{\partial a_2}\right) \quad 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{d}{dt}\left(\frac{\partial S}{\partial a_n}\right) = 0 ;$$

and consequently

$$\frac{\partial^2 S}{\partial a_1 \partial t} + \frac{\partial^2 S}{\partial a_1 \partial q_1}\frac{dq_1}{dt} + \frac{\partial^2 S}{\partial a_1 \partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial^2 S}{\partial a_1 \partial q_n}\frac{dq_n}{dt} = 0,$$

$$\frac{\partial^2 S}{\partial a_2 \partial t} + \frac{\partial^2 S}{\partial a_2 \partial q_1}\frac{dq_1}{dt} + \frac{\partial^2 S}{\partial a_2 \partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial^2 S}{\partial a_2 \partial q_n}\frac{dq_n}{dt} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{\partial^2 S}{\partial a_n \partial t} + \frac{\partial^2 S}{\partial a_n \partial q_1}\frac{dq_1}{dt} + \frac{\partial^2 S}{\partial a_n \partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial^2 S}{\partial a_n \partial q_n}\frac{dq_n}{dt} = 0 \quad (8\cdot71)$$

From the partial differential equation (8·68), which, by hypothesis, S satisfies, we get on differentiating with respect to a_1

$$0 = \frac{\partial^2 S}{\partial a_1 \partial t} + \frac{\partial H}{\partial\left(\dfrac{\partial S}{\partial q_1}\right)}\frac{\partial^2 S}{\partial a_1 \partial q_1} + \frac{\partial H}{\partial\left(\dfrac{\partial S}{\partial q_2}\right)}\frac{\partial^2 S}{\partial a_1 \partial q_2} + \cdots$$

$$+ \frac{\partial H}{\partial\left(\dfrac{\partial S}{\partial q_n}\right)}\frac{\partial^2 S}{\partial a_1 \partial q_n},$$

or, remembering equations (8·7 B),

$$0 = \frac{\partial^2 S}{\partial a_1 \partial t} + \frac{\partial^2 S}{\partial a_1 \partial q_1}\frac{\partial H}{\partial \pi_1} + \frac{\partial^2 S}{\partial a_1 \partial q_2}\frac{\partial H}{\partial \pi_2} + \cdots + \frac{\partial^2 S}{\partial a_1 \partial q_n}\frac{\partial H}{\partial \pi_n},$$

to which we may add similar equations derived by differentiating with respect to $a_2, a_3, \ldots a_n$, namely

$$0 = \frac{\partial^2 S}{\partial a_2 \partial t} + \frac{\partial^2 S}{\partial a_2 \partial q_1}\frac{\partial H}{\partial \pi_1} + \frac{\partial^2 S}{\partial a_2 \partial q_2}\frac{\partial H}{\partial \pi_2} + \cdots + \frac{\partial^2 S}{\partial a_2 \partial q_n}\frac{\partial H}{\partial \pi_n},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$0 = \frac{\partial^2 S}{\partial a_n \partial t} + \frac{\partial^2 S}{\partial a_n \partial q_1}\frac{\partial H}{\partial \pi_1} + \frac{\partial^2 S}{\partial a_n \partial q_2}\frac{\partial H}{\partial \pi_2} + \cdots$$

$$+ \frac{\partial^2 S}{\partial a_n \partial q_n}\frac{\partial H}{\partial \pi_n} \quad (8\cdot711)$$

If now the equations (**8·71**) are solved for $\dfrac{dq_1}{dt}, \dfrac{dq_2}{dt}, \ldots \dfrac{dq_n}{dt}$

and (**8·711**) for $\dfrac{\partial H}{\partial \pi_1}, \dfrac{\partial H}{\partial \pi_2}, \ldots \dfrac{\partial H}{\partial \pi_n}$, we see at once that

$$\frac{dq_1}{dt} = \frac{\partial H}{\partial \pi_1},$$

$$\frac{dq_2}{dt} = \frac{\partial H}{\partial \pi_2} \qquad \cdots \cdots \quad (8\cdot712)$$

$$\cdots \cdots \cdots$$

$$\frac{dq_n}{dt} = \frac{\partial H}{\partial \pi_n}.$$

In order to complete the proof of the theorem, we have still to show that

$$\frac{d\pi_1}{dt} = -\frac{\partial H}{\partial q_1},$$

$$\frac{d\pi_2}{dt} = -\frac{\partial H}{\partial q_2} \qquad \cdots \cdots \quad (8\cdot713)$$

$$\cdots \cdots \cdots$$

$$\frac{d\pi_n}{dt} = -\frac{\partial H}{\partial q_n}.$$

By (**8·7** B) we have

$$\frac{d\pi_1}{dt} = \frac{d}{dt}\frac{\partial S}{\partial q_1}$$

therefore

$$\frac{d\pi_1}{dt} = \frac{\partial^2 S}{\partial t \partial q_1} + \frac{\partial^2 S}{\partial q_1 \partial q_1}\frac{dq_1}{dt} + \frac{\partial^2 S}{\partial q_2 \partial q_1}\frac{dq_2}{dt} + \ldots + \frac{\partial^2 S}{\partial q_n \partial q_1}\frac{dq_n}{dt},$$

or, using (**8·712**),

$$\frac{d\pi_1}{dt} = \frac{\partial^2 S}{\partial t \partial q_1} + \frac{\partial^2 S}{\partial q_1 \partial q_1}\frac{\partial H}{\partial \pi_1} + \frac{\partial^2 S}{\partial q_2 \partial q_1}\frac{\partial H}{\partial \pi_2} + \ldots$$

$$+ \frac{\partial^2 S}{\partial q_n \partial q_1}\frac{\partial H}{\partial \pi_n} \quad (8\cdot714)$$

On the other hand we get by differentiating (**8·68**) partially with respect to q_1, and remembering that the partial differentiation of H with respect to q_1 is not merely what we represent as $\dfrac{\partial H}{\partial q_1}$ in which the p's, i.e. the $\dfrac{\partial S}{\partial q}$'s, are treated as independent

variables but takes account of the q's contained in the $\dfrac{\partial S}{\partial q}$'s,

$$\frac{\partial^2 S}{\partial q_1 \partial t} + \frac{\partial H}{\partial\left(\dfrac{\partial S}{\partial q_1}\right)} \frac{\partial^2 S}{\partial q_1 \partial q_1} + \frac{\partial H}{\partial\left(\dfrac{\partial S}{\partial q_2}\right)} \frac{\partial^2 S}{\partial q_1 \partial q_2} + \dots$$

$$+ \frac{\partial H}{\partial\left(\dfrac{\partial S}{\partial q_n}\right)} \frac{\partial^2 S}{\partial q_1 \partial q_n} + \frac{\partial H}{\partial q_1} = 0.$$

Hence by (**8·7** B),

$$\frac{\partial^2 S}{\partial q_1 \partial t} + \frac{\partial H}{\partial \pi_1} \frac{\partial^2 S}{\partial q_1 \partial q_1} + \frac{\partial H}{\partial \pi_2} \frac{\partial^2 S}{\partial q_1 \partial q_2} + \dots$$

$$+ \frac{\partial H}{\partial \pi_n} \frac{\partial^2 S}{\partial q_1 \partial q_n} + \frac{\partial H}{\partial q_1} = 0 \quad . \ (\mathbf{8 \cdot 715})$$

On comparing (**8·714**) and (**8·715**) we find

$$\frac{d\pi_1}{dt} = - \frac{\partial H}{\partial q_1},$$

and we can establish the validity of the remaining equations (**8·713**) in a similar way. The theorem thus proved was first given by Jacobi (*Vorlesungen ü. Dynamik*, No. **XX**).

We have seen (**8·65**) that

$$S = \int (T - V)dt = \int (2T - E)dt.$$

Therefore

$$S = A - \int E dt$$

by (**8·633**).

If E is constant (conservative system)

$$S = A - E \times \text{time} \quad . \quad . \quad . \quad . \ (\mathbf{8 \cdot 72})$$

and

$$\frac{\partial S}{\partial t} = - E, \ \frac{\partial S}{\partial q_a} = \frac{\partial A}{\partial q_a}.$$

Let us take the constant a_1 (**8·7** A) to be E; then Hamilton's differential equation (**8·68**) becomes

$$H\left(\frac{\partial A}{\partial q_a}, \ q_a\right) = a_1 \quad . \quad . \quad . \quad . \ (\mathbf{8 \cdot 73})$$

From (**8·72**) we get

$$\frac{\partial S}{\partial \alpha_1} = \frac{\partial A}{\partial \alpha_1} - t$$

or

$$\beta_1 = \frac{\partial A}{\partial \alpha_1} - t, \ \text{by} \ (\mathbf{8 \cdot 7} \ B)$$

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and therefore Jacobi's theorem when applied to (**8·73**) takes the form

$$\frac{\partial A}{\partial \alpha_1} = t + \beta_1, \qquad \frac{\partial A}{\partial q_1} = \pi_1,$$

$$\frac{\partial A}{\partial \alpha_2} = \beta_2, \qquad \frac{\partial A}{\partial q_2} = \pi_2,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{(A)} \qquad \cdot \quad \cdot \quad \cdot \quad \text{(B)}$$

$$\frac{\partial A}{\partial \alpha_n} = \beta_n, \qquad \frac{\partial A}{\partial q_n} = \pi_n \quad \cdot \quad \cdot \quad \cdot \quad (\mathbf{8·8})$$

A being a complete integral of (**8·73**) and the π's being identical with the corresponding generalized momenta.

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CHAPTER VI

WAVE PROPAGATION

§ 9. Waves with Unvarying Amplitude

A SIMPLE example of wave motion can be exhibited on a long cord stretched between two fixed points. If one end of the cord be given a sudden jerk and then left fixed the resulting deformation will travel along it towards the other end. Such a deformation is propagated without change of shape, to a first approximation at any rate, and with a constant velocity. Suppose the undisturbed cord to coincide with the X axis, and the disturbance to be travelling in that direction. Let ψ (Fig. 9) represent the ordinates, or displacements, which constitute the
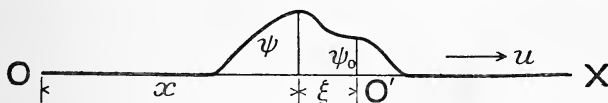
Fig. 9

deformation, and which we shall suppose are all in the same plane. The shape of the disturbance may be represented by

$$\psi = f(\xi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where the abscissa, ξ, corresponding to the ordinate ψ, is measured from a point, $0'$, *which travels with the disturbance*, and where, for convenience, we are taking its positive direction to be opposite to that of the X axis, since the successive displacements, ψ, will then reach an observer at some fixed point on the X axis in the order of increasing values of ξ. The function f is quite arbitrary, depending on the initial disturbance. If x be used to represent the distance, measured in the X direction, of the ordinate ψ from some *fixed* origin, 0,

$$x = (00') - \xi,$$

and if we measure the time from the instant when $0'$ coincides with 0, so that $(00') = ut$, u being the velocity of propagation, then

$$\xi = ut - x$$

and

$$\psi = f(ut - x) \quad . \quad . \quad . \quad . \quad . \quad (9 \cdot 01)$$

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A special and very important case of (**9·01**) is that in which f is a simple harmonic function; for example

$$\psi = A \cos a(ut - x) \quad . \quad . \quad . \quad (9\text{·}011)$$

where A and a are constants. If we define another constant ω by

$$\omega = au,$$

we may give (**9·011**) the form

$$\psi = A \cos \omega\left(t - \frac{x}{u}\right) \quad . \quad . \quad . \quad (9\text{·}012)$$

so that at a fixed point on the cord

$$\psi = A \cos (\omega t - \text{const.}) \quad . \quad . \quad . \quad (9\text{·}013)$$

The period of vibration, τ, will be

$$\tau = \frac{2\pi}{\omega},$$

since the values of ψ will be repeated if t is increased by any integral multiple of $\frac{2\pi}{\omega}$.

At a *given time* the values of ψ at various points, x, will be expressed by

$$\psi = A \cos \left(\text{const.} - \frac{\omega x}{u}\right) \quad . \quad . \quad (9\text{·}014)$$

and it will be seen that the values of ψ repeat themselves over intervals, λ, where

$$\lambda = \frac{2\pi u}{\omega}.$$

The distance, λ, is called the **wave length.** We see that

$$\lambda = u\tau,$$

and we may express (**9·012**) in the form

$$\left.\begin{aligned} \psi &= A \cos 2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right), \\ \psi &= A \cos 2\pi\left(\frac{x}{\lambda} - \frac{t}{\tau}\right). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9\text{·}02)$$

or

A is called the **amplitude,** and the argument of the cosine is called the **phase.** It is clear that we may add any constant to the phase, since it would merely amount to the same thing as a change in the zero from which x or t is measured.

It is an essential feature of wave equations that the dependent variable, ψ, is a function of more than one independent variable. In the example just given there are two such variables, x and t. If we wish to eliminate the particular function, f, in (**9·01**) for

example, we shall have to differentiate with respect to these
independent variables, and so we shall obtain a partial differential
equation, which, since it does not contain the particular function,
f, will include every kind of disturbance travelling along the
cord with a constant velocity u, and without change of shape.
We shall use the abbreviations

$$\frac{df(\xi)}{d\xi} = f' \text{ and } \frac{d^2f(\xi)}{d\xi^2} = f''.$$

Differentiating (**9·01**) partially with respect to t and x, we get

$$\frac{\partial \psi}{\partial t} = uf',$$

and

$$\frac{\partial \psi}{\partial x} = -f'.$$

Therefore

$$\frac{\partial \psi}{\partial t} + u\frac{\partial \psi}{\partial x} = 0 \quad \ldots \quad \text{(9·03)}$$

For a given value of the constant u this equation will not include
among its solutions any representing a propagation in the nega-
tive direction of X. To get a differential equation which includes
both directions of propagation we may either multiply (**9·03**)
by the conjugate equation

$$\frac{\partial \psi}{\partial t} - u\frac{\partial \psi}{\partial x} = 0 \quad \ldots \quad \text{(9·031)}$$

thus obtaining

$$\left(\frac{\partial \psi}{\partial t}\right)^2 = u^2\left(\frac{\partial \psi}{\partial x}\right)^2. \quad \ldots \quad \text{(9·04)}$$

or we may form the second differential quotients,

$$\frac{\partial^2 \psi}{\partial t^2} = u^2f'',$$

$$\frac{\partial^2 \psi}{\partial x^2} = f'',$$

which give the equation

$$\frac{\partial^2 \psi}{\partial t^2} = u^2\frac{\partial^2 \psi}{\partial x^2} \quad \ldots \quad \text{(9·05)}$$

This latter is in fact the equation we arrive at on applying
the principles of mechanics to the motion of a stretched cord,
provided we restrict our attention to small displacements. Let
the stretching force be F and the mass of the cord per unit
length be m and consider a short element of the cord (ab)

(Fig. 9·01) of length l.   At the end a there will be a force with a downward component equal to

$$F\frac{\partial\psi}{\partial l}.$$

If the slope $\frac{\partial\psi}{\partial l}$ is small we may take this downward component to be

$$F\frac{\partial\psi}{\partial x}.$$

At the other end, b, of the element there will be a force the *upward* component of which is

$$F\frac{\partial\psi}{\partial x} + \frac{\partial}{\partial x}\Big(F\frac{\partial\psi}{\partial x}\Big)l,$$

and consequently the component in an upward direction of the resultant force on the element will be

$$Fl\frac{\partial^2\psi}{\partial x^2}.$$
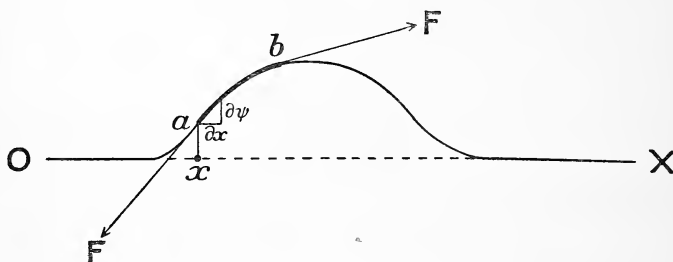
FIG. 9·01

This must be equal to the mass of the element multiplied by its vertical acceleration, namely

$$ml\frac{\partial^2\psi}{\partial t^2},$$

and on equating the two expressions we get

$$\frac{\partial^2\psi}{\partial t^2} = \frac{F}{m}\frac{\partial^2\psi}{\partial x^2} \quad . \quad . \quad . \quad . \quad (9\text{·}051)$$

This equation becomes identical with (9·05) if

$$u = \pm\sqrt{\frac{F}{m}} \quad . \quad . \quad . \quad . \quad (9\text{·}052)$$

and (9·01) is one of its solutions.   We learn therefore that a transverse wave is propagated along the cord with the velocity given by (9·052), provided the slope, $\frac{\partial\psi}{\partial x}$, is everywhere small.

It is instructive to study the transverse motions of a stretched cord in some detail. Confining our attention to motions in one plane, we may represent the arbitrarily given initial configuration by

$$\psi = \psi_0 = f(x),$$

and the initial velocities at different points on the cord by

$$\frac{\partial \psi}{\partial t} = \left(\frac{\partial \psi}{\partial t}\right)_0 = F(x).$$

If the ends of the cord be fixed and if the distance between them be L, the functions $f(x)$ and $F(x)$ will both be zero for $x = 0$ and $x = L$, and moreover ψ and $\frac{\partial \psi}{\partial t}$ will be zero at all times at the points $x = 0$ and $x = L$. We are given then

$$\psi_0 = f(x)$$

$$\left(\frac{\partial \psi}{\partial t}\right)_0 = F(x) \quad . \quad . \quad . \quad . \quad . \quad (9\cdot06)$$

$$\psi = 0 \Big\} \text{ for all values of } t$$
$$\text{and consequently } \frac{\partial \psi}{\partial t} = 0 \Big\} \begin{array}{l} \text{when} \quad x = 0 \\ \text{or when} \quad x = L. \end{array}$$

We shall term the equations **(9·06)** the **boundary conditions**.

Whatever form of solution we adopt, it must not only satisfy the differential equation **(9·05)** or **(9·051)**, but must also conform to the boundary conditions. Such a solution is the following :

$$\psi = \tfrac{1}{2}f(x + ut) + \tfrac{1}{2}f(x - ut) + \frac{1}{2u} \int\limits_{\xi\,=\,x\,-\,ut}^{\xi\,=\,x\,+\,ut} F(\xi)d\xi \quad . \quad (9\cdot07)$$

It satisfies the differential equation, because it is a sum of functions of $x + ut$ and $x - ut$ each of which separately satisfies it, and it is a property of *linear* differential equations, i.e. equations in which powers of the differential quotients higher than the first, or products of the differential quotients, are absent, that the sum of two or more solutions is itself a solution of such an equation. It also satisfies the boundary conditions, since if we give t the value zero the limits of the integral in **(9·07)** become equal to one another and it therefore vanishes, while the rest of the expression becomes

$$\psi_0 = \tfrac{1}{2}f(x) + \tfrac{1}{2}f(x) = f(x).$$

At the same time

$$\left(\frac{\partial \psi}{\partial t}\right)_0 = F(x).$$

To show this let us put the integral in the form

$$\int\limits_{\xi=x-ut}^{\xi=x+ut} F(\xi)d\xi = R(x + ut) - R(x - ut),$$

where R has the property

$$\frac{dR(\xi)}{d\xi} = F(\xi).$$

We easily find that

$$\frac{\partial \psi}{\partial t} = \frac{u}{2}f'(x + ut) - \frac{u}{2}f'(x - ut)$$

$$+ \frac{1}{2u}\left\{ u\frac{dR(x + ut)}{d(x + ut)} + u\frac{dR(x - ut)}{d(x - ut)} \right\}$$

or

$$\frac{\partial \psi}{\partial t} = \frac{u}{2}\{f'(x + ut) - f'(x - ut)\} + \tfrac{1}{2}\{F(x + ut) + F(x - ut)\}$$

$$= F(x) \text{ when } t = 0.$$

The solution (9·07) is usually ascribed to d'Alembert. His contribution to the subject however consisted in showing that any solution of (9·05) must be contained in the expression

$$\psi = f(x + ut) + \phi(x - ut),$$

(*Mémoires de l'académie de Berlin*, 1747). It was actually Euler who first gave the solution in the form (9·07).
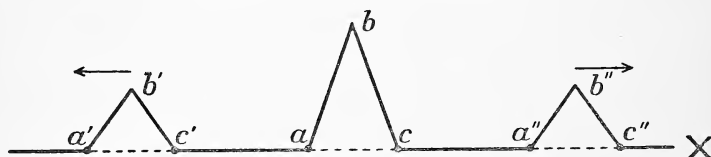
FIG. 9·02

As a simple illustration of the application of d'Alembert's solution (or Euler's solution) let us take the case of a long cord in which displacements are produced at some instant, which we may take to be zero, over a short or limited part of the cord (*a b c*, Fig. 9·02). And let us further suppose that at this instant the velocities are zero. We have therefore

$$\psi_0 = f(x)$$

where f describes the shape of the curve $a\ b\ c$ (Fig. 9·02) and

$$\left(\frac{\partial \psi}{\partial t}\right)_0 = F(x) = 0.$$

Therefore $f(x)$ differs from zero for values of x between a and c

(Fig. 9·02) and is zero for all other values of x, while $F(x)$ is zero for all values of x.   Equation (9·07) now becomes

$$\psi = \tfrac{1}{2}f(x + ut) + \tfrac{1}{2}f(x - ut)$$

which shows that the deformation $a\ b\ c$ splits up into two portions, $a'\ b'\ c'$ and $a''\ b''\ c''$, differing from the initial deformation in having their corresponding ordinates half the original height.   These are propagated in opposite directions with the velocity u.

In using d'Alembert's solution (9·07) we are confronted with the difficulty that while $f(x)$ and $F(x)$ are defined for values of x between 0 and L, nothing seems to be laid down for the behaviour of these functions outside the range of values 0 to L. Yet we need to know how they behave for *any* real value of the independent variable, since in (9·07) the values of the independent variable in the function, f, are $x + ut$ and $x - ut$ and they also range between these limits in the integral $\displaystyle\int F(\xi)d\xi$.

The answer to the question thus raised is contained in the last of the conditions (9·06); but we shall defer it until we have studied an entirely different solution of the differential equation (9·05), and the problem of the vibrating cord, given in 1753 by Daniel Bernoulli.

Bernoulli's method consists in finding particular solutions of the differential equation, each of which is a product of a function of x only and a function of t only.   Thus

$$\psi_1 = X_1 T_1,$$
$$\psi_2 = X_2 T_2,$$
$$\psi_3 = X_3 T_3 \quad . \quad . \quad . \quad . \quad . \quad (9\cdot08)$$
$$. \quad . \quad . \quad .$$
$$\psi_s = X_s T_s,$$
$$. \quad . \quad . \quad .$$

where X_s is a function of x_s only and T_s is a function of t only. Substituting any one of these in the differential equation we have

$$X_s \frac{d^2 T_s}{dt^2} = u^2 T_s \frac{d^2 X_s}{dx^2},$$

and on dividing by the product $X_s T_s$,

$$\frac{1}{T_s} \frac{d^2 T_s}{dt^2} = \frac{u^2}{X_s} \frac{d^2 X}{dx^2} .$$

To satisfy this equation we must equate both sides to the same constant.   Therefore

$$\frac{1}{T_s} \frac{d^2 T_s}{dt^2} = m_s,$$
$$\frac{u^2}{X_s} \frac{d^2 X_s}{dt^2} = m_s,$$

where m_s is any constant. For a reason which will become obvious as we proceed, we chose solutions for which m_s is real and *negative*. We shall therefore write

$$m_s = -\omega_s{}^2$$

where ω_s is real. Consequently we find

$$T_s = A_s \cos \omega_s t + B_s \sin \omega_s t$$

and
$$X_s^i = M_s \cos \frac{\omega_s}{u} x + N_s \sin \frac{\omega_s}{u} x.$$

A_s, B_s, M_s and N_s are constants of integration and we may without any loss of generality take ω_s to be positive. A solution of (9·05) is therefore

$$\psi_s = (A_s \cos \omega_s t + B_s \sin \omega_s t)\left(M_s \cos \frac{\omega_s}{u}x + N_s \sin \frac{\omega_s}{u}x \right) \quad \text{(9·081)}$$

and we can make it satisfy the last of the conditions (9·06), namely $\psi = 0$ at all times when $x = 0$ or $x = L$, if we make $M_s = 0$ and $\dfrac{\omega_s}{u} = \dfrac{s\pi}{L}$, s being a positive integer. Equation (9·081) thus becomes

$$\psi_s = (A_s \cos \omega_s t + B_s \sin \omega_s t) \sin \frac{\omega_s}{u}x . \quad . \text{ (9·082)}$$

in which $A_s N_s$ and $B_s N_s$ (of 9·081) have been denoted by A_s and B_s. In consequence of the property of linear differential equations, which has been described above in connexion with d'Alembert's solution,

$$\psi = \Sigma \psi_s$$

or
$$\Sigma(A_s \cos \omega_s t + B_s \sin \omega_s t) \sin \frac{\omega_s}{u}x \quad . \quad . \quad . \text{ (9·09)}$$

is also a solution of the differential equation and it satisfies the conditions at the ends of the cord. We shall suppose the summation to extend over all positive integral values of s.

Since
$$\omega_s = s\frac{\pi u}{L},$$

$$= \frac{s\pi}{L}\sqrt{\frac{F}{m}},$$

we have for the corresponding period, $\tau_s\left(= \dfrac{2\pi}{\omega_s} \right)$,

$$\tau_s = \frac{1}{s}.2L\sqrt{\frac{m}{F}},$$

and for the frequency

$$\nu_s = s \cdot \frac{1}{2L} \sqrt{\frac{F}{m}},$$

so that Bernoulli's solution represents the state of motion of the cord as a superposition of simple harmonic vibrations, the frequencies of which are integral multiples of a fundamental frequency

$$\nu = \frac{1}{2L} \sqrt{\frac{F}{m}}.$$

It is an interesting historical fact, with which Bernoulli was doubtless acquainted, that Dr. Brook Taylor (*Methodus Incrementorum*, 1715) found that a stretched cord could vibrate according to the law

$$\psi = A \cos \omega t \sin \frac{\omega}{u} x$$

$$= A \cos \frac{s\pi}{L} \sqrt{\frac{F}{m}} t \sin \frac{s\pi}{L} x,$$

where s is any positive integer. Bernoulli was led to the more general expression (9·09) by the physical observation that the fundamental note and its harmonics may be heard simultaneously when a cord is vibrating.

The problem of determining the coefficients A_s and B_s so as to satisfy the initial conditions was not solved till the year 1807 when Fourier showed how an arbitrary function may be expanded as a sum of cosine and sine terms. If in (9·09) we make $t = 0$ we have

$$f(x) = \Sigma A_s \sin \frac{\omega_s}{u} x,$$

and we can determine the coefficients A_s by the methods of § 4, since $f(x)$ is given between the limits $x = 0$ and $x = L$. Similarly if we differentiate ψ partially with respect to t we obtain

$$\frac{\partial \psi}{\partial t} = \Sigma(-\omega_s A_s \sin \omega_s t + \omega_s B_s \cos \omega_s t) \sin \frac{\omega_s}{u} x,$$

and on making $t = 0$,

$$\left(\frac{\partial \psi}{\partial t}\right)_0 = F(x) = \Sigma \omega_s B_s \sin \frac{\omega_s}{u} x,$$

from which Fourier's method enables us to determine the $\omega_s B_s$ and hence the coefficients B_s themselves.

The difficulty which appeared in connexion with d'Alembert's solution does not arise at all in the Bernoulli-Fourier solution

of the problem. If in the Fourier expansions we substitute values of x outside the limits 0 to L, we find

$$f(x) = -f(-x),$$
$$f(L-x) = -f(L+x), \quad . \quad . \quad . \quad . \quad \text{(9·091)}$$
$$F(x) = -F(-x),$$
$$F(L-x) = -F(L+x).$$

This suggests that in d'Alembert's solution we should adopt

$$\psi(x) = -\psi(-x),$$
$$\psi(L-x) = -\psi(L+x),$$
$$\left(\frac{\partial\psi}{\partial t}\right)_x = -\left(\frac{\partial\psi}{\partial t}\right)_{-x} \quad . \quad . \quad . \quad . \quad \text{(9·092)}$$
$$\left(\frac{\partial\psi}{\partial t}\right)_{L-x} = -\left(\frac{\partial\psi}{\partial t}\right)_{L+x}$$

If we do this and imagine the cord extended (Fig. 9·03) both ways beyond the points 0 and L to $-L$ and $2L$, it is obvious that the points 0 and L on the cord must remain undisplaced and the motion of the part between 0 and L will be precisely the same as if these two points had been fixed.

As an illustration suppose

$$f(x) = \varepsilon x, \qquad 0 \leqq x \leqq L/2,$$
$$f(x) = \varepsilon(L-x), \qquad \frac{L}{2} \leqq x \leqq L,$$

where ε is a small positive constant (see Fig. 9·03), and assume the initial velocities to be zero, i.e. $F(x) = 0$.
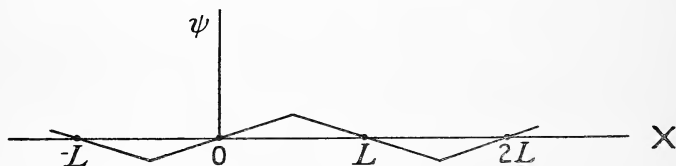
FIG. 9·03

The appropriate Fourier expansion (see § 4·1) is easily found to be

$$f(x) = \frac{4\varepsilon L}{\pi^2}\left(\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi x}{L} + \frac{1}{5^2}\sin\frac{5\pi}{L}x - \ldots\right).$$

Therefore
$$A_1 = \frac{4\varepsilon L}{\pi^2},$$
$$A_2 = 0,$$
$$A_3 = -\frac{4\varepsilon L}{3^2\pi^2},$$

and so on. The coefficients B_s are all zero, and we have

$$\frac{\omega_1}{u} = \frac{\pi}{L},$$

$$\frac{\omega_3}{u} = \frac{3\pi}{L},$$

$$\frac{\omega_5}{u} = \frac{5\pi}{L};$$

therefore

$$\psi = \frac{4\varepsilon L}{\pi^2}\left\{\cos\frac{\pi u}{L}t\sin\frac{\pi}{L}x - \frac{1}{3^2}\cos 3\frac{\pi u}{L}t\sin\frac{3\pi}{L}x\right.$$

$$\left. + \frac{1}{5^2}\cos 5\frac{\pi u}{L}t\sin\frac{5\pi}{L}x - + \ldots\right\}\quad . \quad . \quad (9\cdot092)$$

and the motion is a superposition of simple harmonic vibrations the frequencies of which are odd multiples of the fundamental frequency $\frac{1}{2L}\sqrt{\frac{F}{m}}$. The absence of even multiples is due of course to the special choice of initial conditions.

§ 9·1. Waves with Varying Amplitude

The type of wave represented by equation (9·01), which we may term a one-dimensional wave, since there is only one spacial independent variable involved in its description, is propagated without change in shape or magnitude. We shall now study two other types of one-dimensional wave. These are also propagated without change in shape ; but they become more and more reduced in magnitude the further they travel. If the values of ψ at a given position, x, are plotted against the time, the *shape* of the graph is the same for all positions, x, but the bigger x is, the smaller is the biggest of the ordinates ψ. The first of these is represented by

$$\psi = \frac{1}{x}f(ut - x)\quad . \quad . \quad . \quad . \quad . \quad (9\cdot1)$$

If we slightly extend the use of the term **amplitude,** we may say that the amplitude of this wave is inversely proportional to the distance it has travelled from the origin, $x = 0$. Writing the equation in the form

$$x\psi = f(ut - x),$$

and referring to (9·01) and (9·05), we see that the corresponding partial differential equation is

$$\frac{\partial^2(x\psi)}{\partial t^2} = u^2\frac{\partial^2(x\psi)}{\partial x^2}\quad . \quad . \quad . \quad (9\cdot101)$$

This is equivalent to

$$x\frac{\partial^2\psi}{\partial t^2} = u^2\frac{\partial}{\partial x}\Big(\psi + x\frac{\partial\psi}{\partial x}\Big),$$

or

$$\frac{\partial^2\psi}{\partial t^2} = u^2\Big(\frac{2}{x}\frac{\partial\psi}{\partial x} + \frac{\partial^2\psi}{\partial x^2}\Big). \quad . \quad . \quad . \quad (9 \cdot 102)$$

The other type is one in which the amplitude varies exponentially. It is represented by the equation

$$\psi = e^{-ax}f(ut - x) \quad . \quad . \quad . \quad . \quad (9 \cdot 11)$$

where a is a positive constant. On differentiating we get

$$\frac{\partial^2\psi}{\partial x^2} = a^2\psi + 2ae^{-ax}f' + e^{-ax}f''$$

and on eliminating f' and f'' by means of

$$\frac{1}{u}\frac{\partial\psi}{\partial t} = e^{-ax}f',$$

and

$$\frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2} = e^{-ax}f'',$$

we find for the corresponding differential equation

$$\frac{\partial^2\psi}{\partial x^2} = a^2\psi + \frac{2a}{u}\frac{\partial\psi}{\partial t} + \frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2} \quad . \quad . \quad . \quad (9 \cdot 12)$$

This type of differential equation will be encountered in studying the propagation of an electrical disturbance along a cable.

§ 9·2.   Plane and Spherical Waves

The equation (9·01) will also describe a wave propagated in the direction X in a medium, if x, y and z are the rectangular co-ordinates of a point in the medium. Such a wave is called a **plane wave** since ψ has the same value at all points in any plane $x = $ const. We can easily modify the equation so that it will represent a plane wave travelling in any direction in the medium. For this purpose we introduce new axes of co-ordinates X', Y', Z' with the same origin as X, Y, Z (§ 2:2), so that

$$\psi = f\{ut - (lx' + my' + nz')\} \quad . \quad . \quad . \quad (9 \cdot 2)$$

where l, m and n are the cosines of the angles between the direction of propagation, X, and the axes X', Y', Z' respectively. A plane

$$lx' + my' + nz' = \text{const.},$$

at all points in which ψ has the same value at a given time is called a **wave front**. In general we shall use N to represent the direction of propagation, or a normal to the wave front,

and we may drop the dashes in (9·2). We can eliminate the particular function, f, by means of

$$\frac{\partial^2 \psi}{\partial t^2} = u^2 f'',$$

$$\frac{\partial^2 \psi}{\partial x^2} = l^2 f'', \quad \frac{\partial^2 \psi}{\partial y^2} = m^2 f'', \quad \frac{\partial^2 \psi}{\partial z^2} = n^2 f'',$$

and so obtain

$$\frac{\partial^2 \psi}{\partial t^2} = u^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right),$$

or

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} = u^2 \nabla^2 \psi} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9 \cdot 21)$$

since

$$l^2 + m^2 + n^2 = 1.$$

This last equation is of course much more general than the primitive (9·2) from which it has been derived. The following important example will illustrate this. We may suppose ψ to be a quantity which is determined by r the distance from the origin, so that $\psi = $ function (r). We then have

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial x}$$

and since

$$r^2 = x^2 + y^2 + z^2,$$

we have

$$2r \frac{\partial r}{\partial x} = 2x,$$

therefore

$$\frac{\partial r}{\partial x} = \frac{x}{r},$$

and consequently

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial r} \cdot \frac{x}{r}.$$

Differentiating again with respect to x we get

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{x^2}{r^2} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} - \frac{x^2}{r^3} \frac{\partial \psi}{\partial r},$$

and there are similar expressions for $\dfrac{\partial^2 \psi}{\partial y^2}$ and $\dfrac{\partial^2 \psi}{\partial z^2}$. Adding all three equations we find

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r}.$$

Consequently (9·21) becomes

$$\frac{\partial^2 \psi}{\partial t^2} = u^2 \left\{ \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} \right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9 \cdot 22)$$

and reference to (**9·102**) and the equations immediately preceding it, shows that a solution of (**9·22**) is

$$\psi = \frac{1}{r}f(ut - r) \quad . \quad . \quad . \quad . \quad (9·221)$$

This represents a **spherical wave** propagated with the velocity u and having an amplitude inversely proportional to the distance from the origin.

Except in the case of the transverse wave along a cord we have left the character of the dependent variable, ψ, undefined. It may be a scalar or a vector quantity. In the latter case we have three similar equations associated with the three axes X, Y, Z respectively. Under this heading we may usefully study a more general type of equation which we shall meet when investigating the propagation of electromagnetic disturbances, and of the strain produced in an elastic medium. This equation has the form

$$\frac{\partial^2 \psi_x}{\partial t^2} = A\nabla^2 \psi_x + B\frac{\partial}{\partial x}(\text{div } \boldsymbol{\psi}) \quad . \quad . \quad (9·23)$$

and there are of course two others similarly related to the Y and Z axes.

If $$\text{div } \boldsymbol{\psi} = 0$$

we may, provided B is not infinite in such a case, satisfy the equations (**9·23**) by

$$\boldsymbol{\psi} = f\{ut - (lx + my + nz)\},$$

l, m and n being constants and u being equal to $\sqrt{A\,|}$; so that

$$\psi_x = \alpha f,$$
$$\psi_y = \beta f,$$
$$\psi_z = \gamma f,$$

where α, β and γ, which are the cosines of the angles between the direction of $\boldsymbol{\psi}$ and those of the X, Y and Z axes respectively, are also constants. We easily find that

$$\text{div } \boldsymbol{\psi} = -(\alpha l + \beta m + \gamma n)f',$$

and in order that this may vanish, without involving the simultaneous vanishing of f', it is necessary that

$$\alpha l + \beta m + \gamma n = 0,$$

i.e. the scalar product of the vectors (α, β, γ) and (l, m, n) must be zero. This means that the two vectors, one in the direction of $\boldsymbol{\psi}$ and the other in the direction, N, along which the wave travels, are at right angles to one another. Such a wave is called a **transverse wave**. Waves in which the displacements are in the line of propagation are known as **longitudinal waves**.

Turning to the case where **div ψ** is different from zero, let us differentiate the equations (**9·23**) with respect to x, y, and z respectively and add. We thus get

$$\frac{\partial^2\,(\mathbf{div}\,\boldsymbol{\psi})}{\partial t^2} = A\triangledown^2\,(\mathbf{div}\,\boldsymbol{\psi}) + B\triangledown^2\,(\mathbf{div}\,\boldsymbol{\psi})$$

or, if we write

$$D = \mathbf{div}\,\boldsymbol{\psi},$$

$$\frac{\partial^2 D}{\partial t^2} = (A + B)\boldsymbol{\triangledown}^2 D \quad . \quad . \quad . \quad . \quad (9·24)$$

so that the scalar quantity, D, is propagated with the velocity $\sqrt{A + B}$. Consider now any point on the wave front at some instant, and for convenience imagine the axes placed so that the point is on or near the X axis, and so that the direction of propagation is that of the X axis. We may consider any sufficiently restricted part of the wave front in this neighbourhood to be plane, therefore (see the beginning of § 9·2) differential quotients of the components of **ψ** with respect to y and z are zero in such a neighbourhood and D or **div ψ** reduces to $\dfrac{\partial\psi_x}{\partial x}$, or to $\dfrac{\partial\psi_n}{\partial n}$, if n represents distances measured along the direction of propagation.

In (**9·24**) therefore we are concerned only with displacements in the direction of propagation and the equation represents a *longitudinal* wave.

When we differentiate the first of the equations (**9·23**) with respect to y and subtract the result from that due to differentiating the second one with respect to x, we get

$$\frac{\partial^2}{\partial t^2}\!\left(\frac{\partial\psi_y}{\partial x} - \frac{\partial\psi_x}{\partial y}\right) = A\,\triangledown^2\!\left(\frac{\partial\psi_y}{\partial x} - \frac{\partial\psi_x}{\partial y}\right),$$

or

$$\frac{\partial^2\sigma_z}{\partial t^2} = A\,\triangledown^2(\sigma_z) \quad . \quad . \quad . \quad . \quad (9·25)$$

if we represent **curl ψ** by **σ**. And we have, of course, two further equations containing σ_x and σ_y.

Once again let us imagine the axes moved so that some arbitrarily selected point on a wave front is travelling along the X axis at a given instant. Then in its neighbourhood differential quotients of the components of **ψ** with respect to y and z must be zero, and we are left with $\dfrac{\partial\psi_y}{\partial x}$ and $\dfrac{\partial\psi_z}{\partial x}$ only, since $\dfrac{\partial\psi_x}{\partial x}$ does not occur in **σ** = **curl ψ**. The equations (**9·25**) involve there-

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fore only displacements in directions perpendicular to that of propagation and the equation represents a *transverse* wave travelling with the velocity $\sqrt{A}$].

§ 9·3. PHASE VELOCITY AND GROUP VELOCITY

The differential equations in the foregoing paragraphs, e.g. (9·21) and (9·23), represent wave propagations having the characteristic feature that the velocity of propagation is independent of the *form* of the disturbance or deformation which is being propagated. The velocity of a small transverse disturbance produced in a stretched cord, for instance, in no way depends on the function f(§ 9) which describes its shape. Consider now a simple harmonic wave such as that represented by (9·02) which travels with the velocity $u = \lambda/\tau$. It may happen that when τ is given some other value τ' the velocity $u' = \lambda'/\tau'$ differs from λ/τ. This is the case with light waves in material media. There is no unique velocity of propagation for a luminous disturbance. A question both of practical and theoretical importance is the propagation of a **group** of superposed simple harmonic waves having a narrow range of periods extending from τ to $\tau + \triangle\tau$ and a corresponding range of wave lengths from λ to $\lambda + \triangle\lambda$. Let us first consider *two* superposed waves of the same amplitude. The resultant disturbance may be expressed thus

$$\psi = A \cos 2\pi\Big(\frac{t}{\tau} - \frac{x}{\lambda}\Big) + A \cos 2\pi\Big(\frac{t}{\tau'} - \frac{x}{\lambda'}\Big) . \quad . \quad (9\cdot3)$$

where we have written τ' for $\tau + \triangle\tau$ and λ' for $\lambda + \triangle\lambda$. This is equivalent to

$$\psi = 2A \cos 2\pi\Big\{\tfrac{1}{2}\Big(\frac{1}{\tau} - \frac{1}{\tau'}\Big)t - \tfrac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{\lambda'}\Big)x\Big\} \cos 2\pi$$

$$\Big\{\tfrac{1}{2}\Big(\frac{1}{\tau} + \frac{1}{\tau'}\Big)t - \tfrac{1}{2}\Big(\frac{1}{\lambda} + \frac{1}{\lambda'}\Big)x\Big\}.$$

If now $\tau' - \tau(= \triangle\tau)$ and $\lambda' - \lambda(= \triangle\lambda)$ are both very small, then

$$\psi = 2A \cos 2\pi\Big\{\tfrac{1}{2}\triangle\Big(\frac{1}{\tau}\Big)t - \tfrac{1}{2}\triangle\Big(\frac{1}{\lambda}\Big)x\Big\} \cos 2\pi\Big\{\frac{t}{\tau} - \frac{x}{\lambda}\Big\} \quad . \quad (9\cdot301)$$

or $$\psi = A' \cos 2\pi\Big\{\frac{t}{\tau} - \frac{x}{\lambda}\Big\},$$

where $$A' = 2A \cos 2\pi\Big\{\tfrac{1}{2}\triangle\Big(\frac{1}{\tau}\Big)t - \tfrac{1}{2}\triangle\Big(\frac{1}{\lambda}\Big)x\Big\} \quad . \quad (9\cdot302)$$

If we plot the values of ψ at some given instant against x we shall get a curve like that in Fig. 9·3.

We shall refer to the full line as the wave outline. A crest, a, of the wave outline will travel in the X direction with the horizontal velocity $u = \lambda/\tau$, since it is a point where the phase retains the same value, and therefore

$$d\left\{2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right)\right\} = 0,$$

or

$$\frac{dx}{dt} = \lambda/\tau.$$

The velocity $u = \lambda/\tau$ is called the **phase velocity**. It should be noted that the *crest*, a, will become a *trough* of the wave outline if it passes the point c where the variable amplitude
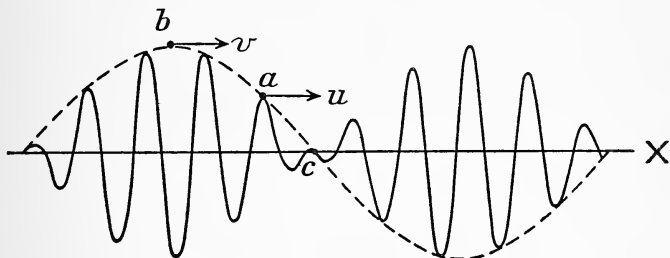
Fig. 9·3

A' (9·302) changes sign. In fact the point, a, will in general travel along the curve represented by the broken line. On the other hand a point, b, on a crest of the broken line will travel with the velocity

$$v = \frac{\triangle\left(\frac{1}{\tau}\right)}{\triangle\left(\frac{1}{\lambda}\right)} \quad . \quad . \quad . \quad . \quad . \quad (9·31)$$

because it is a point where the amplitude A' remains unchanged and for which therefore

$$d\left\{2\pi\left(\triangle\left(\frac{1}{\tau}\right)t - \triangle\left(\frac{1}{\lambda}\right)x\right)\right\} = 0$$

or

$$\triangle\left(\frac{1}{\tau}\right)dt = \triangle\left(\frac{1}{\lambda}\right)dx.$$

This velocity is called the **group velocity**.

We may obviously regard the group velocity as the velocity of propagation of a maximum amplitude and it is clear that, if we have not merely two but any number of simple harmonic

waves superposed on one another they will have a definite group velocity $\dfrac{\triangle 1/\tau}{\triangle 1/\lambda}$ provided the extreme range of periods $\triangle \tau$ is small.

§ 9·4. Dynamics and Geometrical Optics

Hamilton's principal function, S, (**8·65**) plays a part in dynamics like that of the *phase* in wave propagation. The resemblance between the rôles of the two functions—we might almost say their identity—has been so fruitful and suggestive in the recent development of quantum dynamics, that it will be well to study it briefly here.

To begin with we have

$$S = \int (2T - E)dt,$$

or

$$S = \int (p_a \dot{q}_a - E)dt,$$

and consequently

$$S = \int (p_a dq_a - Edt).$$

The simplest case is that in which there is only one degree of freedom and where the potential energy is constant, e.g. a single particle not under the influence of forces, or a body rotating about a fixed axis with no impressed couple acting on it ; so that the energy may be regarded as a function of p only, and during the motion p will remain constant.   In such a case

$$S = pq - Et,$$

or

$$S = px - Et,$$

if, for the present purpose, we use x instead of q for the positional co-ordinate.   On the other hand the phase, in the case of a plane sinusoidal wave (see **9·02**), may be put in the form

$$\phi = 2\pi\left(\frac{x}{\lambda} - \frac{t}{\tau}\right),$$

so that we may think of S, or rather, the product of S and a constant of suitable dimensions, as the phase in a plane sinusoidal wave travelling in the X direction, thus

$$\kappa S = \phi,$$

or

$$\kappa S = 2\pi\left(\frac{x}{\lambda} - \frac{t}{\tau}\right),$$

and therefore

$$\kappa p = \frac{2\pi}{\lambda},$$

$$\kappa E = \frac{2\pi}{\tau},$$

where κ is a constant of suitable dimensions. It is usual to represent $\dfrac{2\pi}{\kappa}$ by h, so that

$$p = \frac{h}{\lambda},$$

$$E = \frac{h}{\tau} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (9·4)$$

$$S = h\left(\frac{x}{\lambda} - \frac{t}{\tau}\right).$$

The **phase velocity** of the wave will evidently be

$$u = \frac{E}{p} \qquad . \quad . \quad . \quad . \quad . \quad (9·41)$$

In classical dynamics there is nothing which enables us to assign a determinate value to κ or h, and moreover the energy, E, involves an arbitrary constant so that u is an arbitrary velocity.

Consider now a small change $\triangle E$ in E and the corresponding small change $\triangle p$ in p. Suppose them to be produced by a force, F, in the case of the particle, or a couple, F, in the case of the rotating body, acting for a short interval of time $\triangle t$, during which it travels (or rotates) the distance (or angle) $\triangle x$. Then we have

$$\triangle E = F \triangle x,$$
$$\triangle p = F \triangle t,$$

and consequently
$$\frac{\triangle x}{\triangle t} = \frac{\triangle E}{\triangle p},$$

or
$$v = \frac{\triangle E}{\triangle p} \qquad . \quad . \quad . \quad . \quad . \quad (9·42)$$

This result (**9·42**) is obviously a special case of the more general equations (**8·43**) given above. It thus appears that the velocity, v, of the particle is identical with the **group velocity** of the corresponding 'mechanical wave'. Unlike the phase velocity this is something quite definite.

The analogy between classical dynamics and wave propagation extends still further. There is a complete correspondence between the principle of least action of Maupertuis (**8·636**) and Fermat's principle in optics. This will be fully explained later. It will suffice at this stage to say that Fermat's principle is the basis of *geometrical* optics, i.e. of optical phenomena in which the wave length of the light is very short in comparison with the dimensions of the optical apparatus, apertures, lenses, etc. In these phenomena the *absolute* value of the wave length is not

of importance, a circumstance which corresponds to the fact, pointed out above, that classical dynamics does not contain anything that enables us to assign a value to the constant h.

Now classical dynamics becomes inadequate when applied to very small systems (electrons, atoms, etc.) and the analogy between it and geometrical optics suggested to Schroedinger that this inadequacy may be of the same kind as that of the principles of geometrical optics when the dimensions of the apparatus or apertures are very small. We shall refer to this assumption as **Schroedinger's Principle** and leave a more complete study of its consequences till a later stage.

It will be recollected that the phase velocity,

$$u = \frac{E}{p},$$

of the 'mechanical wave' of classical dynamics is indeterminate on account of the presence in E of an arbitrary constant. Let us briefly study the consequences of the relativistic hypothesis that the energy of a particle is proportional to its mass, i.e.

$$E = mc^2 \quad . \quad . \quad . \quad . \quad . \quad (9\cdot43)$$

where c is a universal constant with the dimensions of a velocity. We shall have from $(9\cdot42)$

$$v = \frac{\triangle(mc^2)}{\triangle(mv)},$$

therefore

$$v = \frac{\triangle m \cdot c^2}{\triangle m \cdot v + \triangle v \cdot m},$$

or

$$2\frac{\triangle m}{m} = \frac{\triangle\left(\dfrac{v^2}{c^2}\right)}{\left(1 - \dfrac{v^2}{c^2}\right)},$$

and hence $m\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{1}{2}} = \text{constant.}$

This constant is obviously equal to the mass of the particle when its velocity is zero, and if we denote it by m_0 we have

$$m = m_0\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \quad . \quad . \quad . \quad (9\cdot44)$$

for the law of variation of mass with velocity.

Equation $(9\cdot44)$ shows that c is upper limit of velocity for a particle, since if $v = c$ the mass m becomes infinite. It has received a beautiful experimental confirmation by Bucherer who found c to have the same value as the velocity of radiation in empty space.

CHAPTER VII

ELASTICITY

§ 9·5. Homogeneous Strain

THERE is overwhelming evidence for the view that all material media have a granular constitution. They are made of molecules, atoms, electrons and, for anything we know, still smaller particles, which we may be able to recognize in the future. Now when we speak of a volume element, $dx\,dy\,dz$, in a medium, as for example in the theorem of Gauss in § 3, we have in mind a small volume which in the end approaches the limit zero, or to be more precise, dx, dy and dz separately approach the limit zero. We shall, however, make negligible errors when we are concerned with large volumes, or distances, if we suppose dx, dy and dz to approach some very small limit differing from zero. When this small limit is large compared with the distances separating the particles of which the medium is constituted we shall speak of the medium as *continuous*. Let (x, y, z) be the co-ordinates of a point (e.g. the middle point) in a volume element of a continuous medium when in its undisplaced or undeformed condition, and let (α, β, γ) be a displacement (which we shall usually take to be small) of the medium which, in its undisplaced condition, is at the point (x, y, z); then α, β and γ will be functions of x, y and z and the time, t, or

$$
\begin{aligned}
\alpha &= \alpha(x, y, z, t), \\
\beta &= \beta(x, y, z, t) \\
\gamma &= \gamma(x, y, z, t).
\end{aligned}
\qquad \dots \quad (9\text{·}5)
$$

When we are dealing with static conditions we may omit the reference to the time, and equations (9·5) become

$$
\begin{aligned}
\alpha &= \alpha(x, y, z), \\
\beta &= \beta(x, y, z) \\
\gamma &= \gamma(x, y, z).
\end{aligned}
\qquad \dots \quad (9\text{·}501)
$$

In consequence of this displacement, a particle of the medium,

originally at (x, y, z), will have moved to a neighbouring point (ξ, η, ζ), such that

$$\begin{aligned}
\xi &= x + \alpha, \\
\eta &= y + \beta, \\
\zeta &= z + \gamma.
\end{aligned} \qquad \cdots \qquad (9\cdot51)$$

If (x_1, y_1, z_1), $(\alpha_1, \beta_1, \gamma_1)$ and (ξ_1, η_1, ζ_1) refer to a neighbouring particle, we shall have

$$\alpha_1 - \alpha = \frac{\partial \alpha}{\partial x}(x_1 - x) + \frac{\partial \alpha}{\partial y}(y_1 - y) + \frac{\partial \alpha}{\partial z}(z_1 - z) \quad . \quad (9\cdot52)$$

Now it follows from $(9\cdot51)$ that

$$\xi_1 - \xi = x_1 - x + \alpha_1 - \alpha,$$

and we have therefore

$$\xi_1 - \xi = x_1 - x + \frac{\partial \alpha}{\partial x}(x_1 - x) + \frac{\partial \alpha}{\partial y}(y_1 - y) + \frac{\partial \alpha}{\partial z}(z_1 - z) \qquad (9\cdot521)$$

and corresponding expressions for $\eta_1 - \eta$ and $\zeta_1 - \zeta$.

In these equations, $x_1 - x$, $y_1 - y$ and $z_1 - z$ are the X, Y and Z components of a vector $\mathbf{r}$ which specifies the position of one particle, relatively to that of the other, before displacement has occurred. Let $\boldsymbol{\rho}$ be the corresponding vector after displacement. We have therefore

$$\begin{aligned}
r_x &= x_1 - x, \\
r_y &= y_1 - y, \\
r_z &= z_1 - z, \\
\varrho_x &= \xi_1 - \xi, \\
\varrho_y &= \eta_1 - \eta, \\
\varrho_z &= \zeta_1 - \zeta.
\end{aligned} \qquad \cdots \qquad (9\cdot522)$$

From $(9\cdot521)$ and $(9\cdot522)$ we get

$$\begin{aligned}
\varrho_x &= r_x\left(1 + \frac{\partial \alpha}{\partial x}\right) + r_y\frac{\partial \alpha}{\partial y} + r_z\frac{\partial \alpha}{\partial z}, \\
\varrho_y &= r_x\frac{\partial \beta}{\partial x} + r_y\left(1 + \frac{\partial \beta}{\partial y}\right) + r_z\frac{\partial \beta}{\partial z}, \\
\varrho_z &= r_x\frac{\partial \gamma}{\partial x} + r_y\frac{\partial \gamma}{\partial y} + r_z\left(1 + \frac{\partial \gamma}{\partial z}\right) \quad . \quad (9\cdot523)
\end{aligned}$$

It may happen that the displacements (α, β, γ) merely move the medium, or the body which it constitutes, as a whole, i.e. as if it were rigid; but in general the change will consist of such a motion of the body, as a whole, together with some deformation or **strain**.

Instead of considering the point (x, y, z) and *one* neighbouring point (x_1, y_1, z_1), let us consider three neighbouring points

which we shall distinguish by the subscripts 1, 2, and 3. We shall now have three vectors, **r**, namely :

$$\mathbf{r}_1 \equiv (x_1 - x,\, y_1 - y,\, z_1 - z),$$
$$\mathbf{r}_2 \equiv (x_2 - x,\, y_2 - y,\, z_2 - z), \quad . \quad . \quad . \quad (9\cdot53)$$
$$\mathbf{r}_3 \equiv (x_3 - x,\, y_3 - y,\, z_3 - z),$$

in the undisplaced or undeformed state of the medium, which, after displacement become

$$\boldsymbol{\rho}_1 \equiv (\xi_1 - \xi,\, \eta_1 - \eta,\, \zeta_1 - \zeta),$$
$$\boldsymbol{\rho}_2 \equiv (\xi_2 - \xi,\, \eta_2 - \eta,\, \zeta_2 - \zeta), \quad . \quad . \quad (9\cdot531)$$
$$\boldsymbol{\rho}_3 \equiv (\xi_3 - \xi,\, \eta_3 - \eta,\, \zeta_3 - \zeta).$$

The vectors **r** will determine a parallelopiped the volume of which is (§ 2·1)

$$\begin{vmatrix} r_{1x}, & r_{1y}, & r_{1z} \\ r_{2x}, & r_{2y}, & r_{2z} \\ r_{3x}, & r_{3y}, & r_{3z} \end{vmatrix} \quad . \quad . \quad . \quad . \quad (9\cdot532)$$

After displacement this volume will become

$$\begin{vmatrix} \rho_{1x}, & \rho_{1y}, & \rho_{1z} \\ \rho_{2x}, & \rho_{2y}, & \rho_{2z} \\ \rho_{3x}, & \rho_{3y}, & \rho_{3z} \end{vmatrix} \quad . \quad . \quad . \quad . \quad (9\cdot533)$$

If we substitute the expressions in (9·523) for the ρ's in (9·533) we get

$$\begin{vmatrix} r_{1x}\left(1 + \dfrac{\partial\alpha}{\partial x}\right) + r_{1y}\dfrac{\partial\alpha}{\partial y} + r_{1z}\dfrac{\partial\alpha}{\partial z}, & r_{1x}\dfrac{\partial\beta}{\partial x} + r_{1y}\left(1 + \dfrac{\partial\beta}{\partial y}\right) + r_{1z}\dfrac{\partial\beta}{\partial z}, \\ & r_{1x}\dfrac{\partial\gamma}{\partial x} + r_{1y}\dfrac{\partial\gamma}{\partial y} + r_{1z}\left(1 + \dfrac{\partial\gamma}{\partial z}\right), \\[2mm] r_{2x}\left(1 + \dfrac{\partial\alpha}{\partial x}\right) + r_{2y}\dfrac{\partial\alpha}{\partial y} + r_{2z}\dfrac{\partial\alpha}{\partial z}, & r_{2x}\dfrac{\partial\beta}{\partial x} + r_{2y}\left(1 + \dfrac{\partial\beta}{\partial y}\right) + r_{2z}\dfrac{\partial\beta}{\partial z}, \\ & r_{2x}\dfrac{\partial\gamma}{\partial x} + r_{2y}\dfrac{\partial\gamma}{\partial y} + r_{2z}\left(1 + \dfrac{\partial\gamma}{\partial z}\right), \\[2mm] r_{3x}\left(1 + \dfrac{\partial\alpha}{\partial x}\right) + r_{3y}\dfrac{\partial\alpha}{\partial y} + r_{3z}\dfrac{\partial\alpha}{\partial z}, & r_{3x}\dfrac{\partial\beta}{\partial x} + r_{3y}\left(1 + \dfrac{\partial\beta}{\partial y}\right) + r_{3z}\dfrac{\partial\beta}{\partial z}, \\ & r_{3x}\dfrac{\partial\gamma}{\partial x} + r_{3y}\dfrac{\partial\gamma}{\partial y} + r_{3z}\left(1 + \dfrac{\partial\gamma}{\partial z}\right), \end{vmatrix}$$
$$(9\cdot534)$$

which is equal to the product

$$\begin{vmatrix} r_{1x}, & r_{1y}, & r_{1z} \\ r_{2x}, & r_{2y}, & r_{2z} \\ r_{3x}, & r_{3y}, & r_{3z} \end{vmatrix} \times \begin{vmatrix} 1 + \dfrac{\partial\alpha}{\partial x}, & \dfrac{\partial\beta}{\partial x}, & \dfrac{\partial\gamma}{\partial x} \\[2mm] \dfrac{\partial\alpha}{\partial y}, & 1 + \dfrac{\partial\beta}{\partial y}, & \dfrac{\partial\gamma}{\partial y} \\[2mm] \dfrac{\partial\alpha}{\partial z}, & \dfrac{\partial\beta}{\partial z}, & 1 + \dfrac{\partial\gamma}{\partial z} \end{vmatrix} \quad (9\cdot535)$$

as can easily be verified by applying the rule for multiplying determinants.

If the differential quotients $\dfrac{\partial \alpha}{\partial x}$, etc., are very small, so that we may neglect products of two or more of them by comparison with the differential quotients themselves, (9·535) becomes

$$\begin{vmatrix} \varrho_{1x}, & \varrho_{1y}, & \varrho_{1z} \\ \varrho_{2x}, & \varrho_{2y}, & \varrho_{2z} \\ \varrho_{3x}, & \varrho_{3y}, & \varrho_{3z} \end{vmatrix} = \begin{vmatrix} r_{1x}, & r_{1y}, & r_{1z} \\ r_{2x}, & r_{2y}, & r_{2z} \\ r_{3x}, & r_{3y}, & r_{3z} \end{vmatrix} \times \left(1 + \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}\right),$$

or

$$\begin{pmatrix} \text{volume after} \\ \text{displacement} \end{pmatrix} = \begin{pmatrix} \text{original} \\ \text{volume} \end{pmatrix} \times (1 + \mathbf{div}\,(\alpha, \beta, \gamma)) \quad . \ (9\cdot536)$$

and consequently

$$\boxed{\mathbf{div}\,(\alpha, \beta, \gamma) = \frac{\text{Increment in volume}}{\text{per unit volume}}} \quad . \ (9\cdot54)$$

If the body is merely displaced like a rigid body, and *not strained*, this divergence will be zero ; but the converse proposition will not in general be true. We shall call $\mathbf{div}\,(\alpha, \beta, \gamma)$ the **dilatation** of the medium at (x, y, z). It is evident from its physical meaning (9·54) that it is an invariant.

The set of nine quantities

$$\begin{array}{ccc} \dfrac{\partial \alpha}{\partial x}, & \dfrac{\partial \alpha}{\partial y}, & \dfrac{\partial \alpha}{\partial z}, \\[2mm] \dfrac{\partial \beta}{\partial x}, & \dfrac{\partial \beta}{\partial y}, & \dfrac{\partial \beta}{\partial z}, \\[2mm] \dfrac{\partial \gamma}{\partial x}, & \dfrac{\partial \gamma}{\partial y}, & \dfrac{\partial \gamma}{\partial z}, \end{array} \qquad \cdots \cdots \ (9\cdot55)$$

constitutes a tensor of the second rank (§ 2·3). It is convenient to call it the displacement tensor, since *in general* it specifies what may be described as a **pure strain** superposed on a displacement of the body as a whole.

In equations (9·521) let us suppose the origin of the co-ordinates to be shifted to the particle (x, y, z) so that $x = y = z = 0$ and suppose the particle to remain at the origin so that $\xi = \eta = \zeta = 0$. Then

$$\xi_1 = x_1\left(1 + \frac{\partial \alpha}{\partial x}\right) + y_1\frac{\partial \alpha}{\partial y} + z_1\frac{\partial \alpha}{\partial z},$$

and there are two corresponding expressions for η_1 and ζ_1; or, dropping the subscript, 1,

$$\xi = x\left(1 + \frac{\partial\alpha}{\partial x}\right) + y\frac{\partial\alpha}{\partial y} + z\frac{\partial\alpha}{\partial z},$$

$$\eta = x\frac{\partial\beta}{\partial x} + y\left(1 + \frac{\partial\beta}{\partial y}\right) + z\frac{\partial\beta}{\partial z},$$

$$\zeta = x\frac{\partial\gamma}{\partial x} + y\frac{\partial\gamma}{\partial y} + z\left(1 + \frac{\partial\gamma}{\partial z}\right). \quad . \quad . \quad . \quad (9\cdot56)$$

We shall now consider a strain or set of displacements with the property that the components, $\frac{\partial\alpha}{\partial x}$, etc., of the displacement tensor are constants. We may therefore write (9·56) in the form

$$\xi = \lambda_{11}x + \lambda_{12}y + \lambda_{13}z,$$
$$\eta = \lambda_{21}x + \lambda_{22}y + \lambda_{23}z, \quad . \quad . \quad . \quad (9\cdot561)$$
$$\zeta = \lambda_{31}x + \lambda_{32}y + \lambda_{33}z,$$

where the coefficients, λ, are constants. It is clear that, on solving (9·561) for x, y and z, we shall get equations of the form

$$x = \mu_{11}\xi + \mu_{12}\eta + \mu_{13}\zeta,$$
$$y = \mu_{21}\xi + \mu_{22}\eta + \mu_{23}\zeta, \quad . \quad . \quad . \quad (9\cdot562)$$
$$z = \mu_{31}\xi + \mu_{32}\eta + \mu_{33}\zeta,$$

where the coefficients, μ, are likewise constants. Consider now two parallel planes, in the undisplaced medium represented by

$$Ax + By + Cz + D = 0,$$
$$Ax + By + Cz + D_1 = 0 \quad . \quad . \quad . \quad (9\cdot57)$$

After displacement the particles in these planes will be situated in loci, the equations of which we shall obtain by substituting for x, y and z the expressions (9·562). Obviously we shall again obtain linear equations and it will be seen that, in both, the coefficients of ξ, η and ζ are the same, i.e. the equations have the form

$$\Lambda\xi + M\eta + N\zeta + \Omega = 0,$$
$$\Lambda\xi + M\eta + N\zeta + \Omega_1 = 0 \quad . \quad . \quad . \quad (9\cdot571)$$

where Λ, M, N, Ω and Ω_1 are constants. Expressed in words : **particles, which before displacement or strain lie in parallel planes, will lie in parallel planes after displacement.** It follows, since planes intersect in straight lines, that **particles, which in the unstrained condition of the medium lie in parallel straight lines, will also be found to be in**

parallel straight lines in the strained condition of the medium. Such a strain is called a **homogeneous strain.**

§ 9·6. ANALYSIS OF STRAINS

It is clear that a homogeneous strain, as just defined, includes not merely a strain in the stricter sense of the term, i.e. a pure strain, but also, in general, a displacement of the medium or body as a whole. Let us examine what happens to the portion of the continuous medium within the sphere

$$x^2 + y^2 + z^2 = R^2 \quad . \quad . \quad . \quad . \quad (9\cdot6)$$

when subjected to a homogeneous strain, supposing the central point to continue undisplaced, a supposition which does not really entail any loss in generality, since we may, if we desire, imagine the medium to be given a subsequent translation as a whole. On substituting for x, y and z the expressions (**9·562**), we obtain an equation like

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\xi\zeta + 2h\xi\eta = R^2 \ . \ (9\cdot601)$$

where a, b, c, etc., are constants formed from the constants μ in (**9·562**). This must represent an ellipsoid, since the radii vectores $\rho \equiv (\xi, \eta, \zeta)$ are necessarily positive and finite in all directions ; and we may, by altering the directions of the co-ordinate axes, give the equation the simpler form

$$a_0\xi^2 + b_0\eta^2 + c_0\zeta^2 = R^2 \quad . \quad . \quad . \quad (9\cdot602)$$

We conclude therefore that **a pure strain** (if it is homogeneous) **consists in extensions parallel to three lines at right angles to one another.** These three mutually perpendicular lines are called the **principal axes** of the strain and the ellipsoid (**9·601**) or (**9·602**) is called the **strain ellipsoid.** It is perhaps needless to remark that the term *extension* is used algebraically to include *contraction*.

It will be observed that, when the co-ordinate axes are parallel to the principal axes of strain, equations (**9·56**) or (**9·561**) take the form :

$$\xi = x\left(1 + \frac{\partial a}{\partial x}\right),$$

$$\eta = y\left(1 + \frac{\partial \beta}{\partial y}\right), \quad . \quad . \quad . \quad (9\cdot603)$$

$$\zeta = z\left(1 + \frac{\partial \gamma}{\partial z}\right) ;$$

the $\frac{\partial \beta}{\partial x}$, $\frac{\partial \alpha}{\partial y}$, $\frac{\partial \gamma}{\partial y}$, etc., vanishing. Similarly equations (9·562) become

$$x = \frac{\xi'}{1 + \dfrac{\partial \alpha}{\partial x}},$$

$$y = \frac{\eta}{\left(1 + \dfrac{\partial \beta}{\partial y}\right)} \qquad . \quad . \quad . \quad . \quad (9·604)$$

$$z = \frac{\zeta}{\left(1 + \dfrac{\partial \gamma}{\partial z}\right)} \; ;$$

so that the equation (9·602) of the strain ellipsoid is

$$\frac{\xi^2}{\left(1 + \dfrac{\partial \alpha}{\partial x}\right)^2} + \frac{\eta^2}{\left(1 + \dfrac{\partial \beta}{\partial y}\right)^2} + \frac{\zeta^2}{\left(1 + \dfrac{\partial \gamma}{\partial z}\right)^2} = R^2 \quad . \quad (9·605)$$

The components of the tensor 9·55) do not, in general, all vanish even when the medium is not strained at all in the stricter sense of the term. They vanish for a pure translation, since each of the components α, β and γ has the same value at all points (x, y, z). Consider now a *very small* pure rotation, for convenience about an axis through the origin, and represented in magnitude and direction by

$$\mathbf{q} \equiv (q_x, \; q_y, \; q_z).$$

The consequent displacement of a particle, the original position of which is determined by $\mathbf{r} \equiv (x, y, z)$, is (see equation 6·1)

$$\begin{aligned}
\alpha &= q_y z - q_z y, \\
\beta &= q_z x - q_x z, \qquad . \quad . \quad . \quad . \quad (9·61) \\
\gamma &= q_x y - q_y x.
\end{aligned}$$

The q_x, q_y and q_z have of course *the same values for all particles* and are therefore independent of x, y and z. We have consequently

$$\frac{\partial \alpha}{\partial x} = 0, \qquad \frac{\partial \alpha}{\partial y} = -q_z, \quad \frac{\partial \alpha}{\partial z} = q_y,$$

$$\frac{\partial \beta}{\partial x} = q_z, \qquad \frac{\partial \beta}{\partial y} = 0, \qquad \frac{\partial \beta}{\partial z} = -q_x,$$

$$\frac{\partial \gamma}{\partial x} = -q_y, \quad \frac{\partial \gamma}{\partial y} = q_x, \qquad \frac{\partial \gamma}{\partial z} = 0.$$

In this case therefore the tensor (**9·55**) becomes

$$
\begin{array}{ccc}
0, & -\,q_z, & q_y \\
q_z, & 0, & -\,q_x \\
-\,q_y, & q_x, & 0,
\end{array}
\qquad . \quad . \quad . \quad . \quad (9\text{·}62)
$$

all the components being constants. We notice the following relations between them :

$$
\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x} = 0,
$$

$$
\frac{\partial\alpha}{\partial z} + \frac{\partial\gamma}{\partial x} = 0, \quad . \quad . \quad . \quad . \quad . \quad (9\text{·}621)
$$

$$
\frac{\partial\beta}{\partial z} + \frac{\partial\gamma}{\partial y} = 0,
$$

and also that the components of the small rotation $\mathbf{q} \equiv (q_x, q_y, q_z)$ can be expressed in terms of those of the displacement tensor in the following way :

$$
2q_x = \frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z},
$$

$$
2q_y = \frac{\partial\alpha}{\partial z} - \frac{\partial\gamma}{\partial x}, \quad . \quad . \quad . \quad . \quad (9\text{·}622)
$$

$$
2q_z = \frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}.
$$

We see that $\dfrac{\partial\alpha}{\partial x}, \dfrac{\partial\beta}{\partial y}, \dfrac{\partial\gamma}{\partial z}$ and the three quantities represented by the expressions (**9·621**) are *unaffected* by any small displacement of the body as a whole, and therefore their values are determined by the *nature of the strain only*. This suggests that we should seek to describe a pure strain in terms of these six quantities. It is easy to do this. The first of the equations (**9·523**) may be written

$$
\varrho_x = r_x\!\left(1 + \frac{\partial\alpha}{\partial x}\right) + r_y\tfrac{1}{2}\!\left(\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x}\right) + r_z\tfrac{1}{2}\!\left(\frac{\partial\alpha}{\partial z} + \frac{\partial\gamma}{\partial x}\right)
$$
$$
+ r_y\tfrac{1}{2}\!\left(\frac{\partial\alpha}{\partial y} - \frac{\partial\beta}{\partial x}\right) + r_z\tfrac{1}{2}\!\left(\frac{\partial\alpha}{\partial z} - \frac{\partial\gamma}{\partial x}\right),
$$

or, by (**9·622**),

$$
\varrho_x = r_x\!\left(1 + \frac{\partial\alpha}{\partial x}\right) + r_y\tfrac{1}{2}\!\left(\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x}\right) + r_z\tfrac{1}{2}\!\left(\frac{\partial\alpha}{\partial z} + \frac{\partial\gamma}{\partial x}\right)
$$
$$
+ q_y r_z - q_z r_y.
$$

The last part of this expression merely represents a contribution due to the rotation of the body as a whole (**9·61**). The rest is

quite independent of any displacement of the body as a whole and we may therefore describe a pure strain by the equations

$$\varrho_x - r_x = r_x\frac{\partial a}{\partial x} + r_y\tfrac{1}{2}\left(\frac{\partial a}{\partial y} + \frac{\partial \beta}{\partial x}\right) + r_z\tfrac{1}{2}\left(\frac{\partial a}{\partial z} + \frac{\partial \gamma}{\partial x}\right),$$

$$\varrho_y - r_y = r_x\tfrac{1}{2}\left(\frac{\partial \beta}{\partial x} + \frac{\partial a}{\partial y}\right) + r_y\frac{\partial \beta}{\partial y} + r_z\tfrac{1}{2}\left(\frac{\partial \beta}{\partial z} + \frac{\partial \gamma}{\partial y}\right),$$

$$\varrho_z - r_z = r_x\tfrac{1}{2}\left(\frac{\partial \gamma}{\partial x} + \frac{\partial a}{\partial z}\right) + r_y\tfrac{1}{2}\left(\frac{\partial \gamma}{\partial y} + \frac{\partial \beta}{\partial z}\right) + r_z\frac{\partial \gamma}{\partial z} . \quad . \quad (9\cdot63)$$

We shall speak of the symmetrical tensor

$$\frac{\partial a}{\partial x}, \ \tfrac{1}{2}\left(\frac{\partial a}{\partial y} + \frac{\partial \beta}{\partial x}\right), \qquad \tfrac{1}{2}\left(\frac{\partial a}{\partial z} + \frac{\partial \gamma}{\partial x}\right),$$

$$\tfrac{1}{2}\left(\frac{\partial \beta}{\partial x} + \frac{\partial a}{\partial y}\right), \frac{\partial \beta}{\partial y}, \tfrac{1}{2}\left(\frac{\partial \beta}{\partial z} + \frac{\partial \gamma}{\partial y}\right),$$

$$\tfrac{1}{2}\left(\frac{\partial \gamma}{\partial x} + \frac{\partial a}{\partial z}\right), \qquad \tfrac{1}{2}\left(\frac{\partial \gamma}{\partial y} + \frac{\partial \beta}{\partial z}\right), \frac{\partial \gamma}{\partial z} \quad . \quad . \quad (9\cdot64)$$

as the **strain tensor** and represent it by

$$s_{xx}, \ s_{xy}, \ s_{xz},$$
$$s_{yx}, \ s_{yy}, \ s_{yz},$$
$$s_{zx}, \ s_{zy}, \ s_{zz} \quad . \quad . \quad . \quad . \quad (9\cdot641)$$

If we write ξ, η and ζ for the components of $\boldsymbol{\rho}$; x, y and z for those of $\mathbf{r}$ and (a, β, γ) for the difference of these two vectors, i.e.

$$(a, \ \beta, \ \gamma) \equiv (\varrho_x - r_x, \ \varrho_y - r_y, \ \varrho_z - r_z),$$

then equations $(9\cdot63)$ assume the more compact form

$$a = xs_{xx} + ys_{xy} + zs_{xz},$$
$$\beta = xs_{yx} + ys_{yy} + zs_{yz},$$
$$\gamma = xs_{zx} + ys_{zy} + zs_{zz} \quad . \quad . \quad . \quad (9\cdot65)$$

If M represent the scalar product of (a, β, γ) and $\mathbf{r} \equiv (x, y, z)$, we have

$$s_{xx}x^2 + s_{xy}xy + s_{xz}xz$$
$$+ \ s_{yx}yx + s_{yy}y^2 + s_{yz}yz$$
$$+ \ s_{zx}zx + s_{zy}zy + s_{zz}z^2 = M \quad . \quad . \quad . \quad (9\cdot66)$$

In Fig. 9·6 the vector (a, β, γ) is represented by (ab) and the scalar product, M, is therefore equal to the product of $\mathbf{r}$ and (ac), or the product of $\mathbf{r}$ and the component of the displacement (a, β, γ) in the direction of $\mathbf{r}$. The quotient of (ac), by $\mathbf{r}$ is called the **elongation** in the direction of $\mathbf{r}$. The elongation is therefore equal to

$$\frac{(ac)}{\mathbf{r}} = \frac{(ac)\mathbf{r}}{\mathbf{r}^2} = \frac{M}{\mathbf{r}^2} \quad . \quad . \quad . \quad . \quad (9\cdot661)$$

Obviously the elongations in the directions of the co-ordinate axes are $\dfrac{\partial \alpha}{\partial x}$, $\dfrac{\partial \beta}{\partial y}$ and $\dfrac{\partial \gamma}{\partial z}$. The elongations in the directions of the principal axes of the strain are called the **principal elongations.**

If we introduce the principal axes of strain as co-ordinate axes, (9·66) becomes

$$S_{xx}x^2 + S_{yy}y^2 + S_{zz}z^2 = M \quad . \quad . \quad (9\cdot662)$$

in which we have used S_{xx}, S_{yy} and S_{zz} to represent the values which s_{xx}, s_{yy} and s_{zz} respectively assume when these axes are used (s_{xy}, s_{zx}, etc., of course vanish; see the remark after equation 9·603).

If S_{xx}, S_{yy} and S_{zz} in (9·662) are all positive, M must be positive and the locus of all points (x, y, z) for which M has the same positive value is an ellipsoid. All particles, which in their undisplaced condition lie on this ellipsoid, experience an elongation equal to $M/\mathbf{r}^2$ (by equation 9·661). The radial elongation is positive in all directions and inversely proportional to the square of the radius vector $\mathbf{r}$. On the other hand if S_{xx}, S_{yy} and S_{zz} are all negative the radial elongation will be negative in all directions. When S_{xx}, S_{yy} and S_{zz} have not all the same sign, the locus of the points (x, y, z) for which M has the same value will be an hyperboloid. This hyperboloid and its conjugate, obtained by giving M the same numerical value, but with the opposite sign, will represent the elongation of the medium in all directions; and here it should be remarked that it is of no consequence (so far as the elongation is concerned) what the absolute value of M may be, since the elongation in a given direction is the same for all particles when the strain is homogeneous and pure. To see that this is the case divide both sides of (9·662) by $\mathbf{r}^2$. We obtain
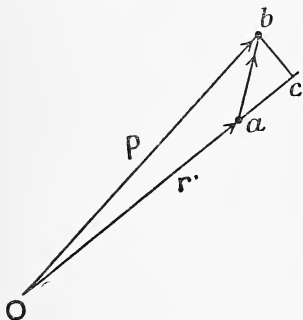
FIG. 9·6

$$S_{xx}l^2 + S_{yy}m^2 + S_{zz}n^2 = \text{elongation};$$

therefore for given values of l, m and n, i.e. for a given direction the elongation is constant. We may therefore just as well assign to M the *absolute* value 1, and the locus (and its conjugate, if it has one) is called the **elongation quadric.**

The direction cosines of the normal at a point on the surface

(**9·66**) or (**9·662**) are proportional to the components α, β and γ of the corresponding displacement.

The conjugated elongation quadrics, or the two families of surfaces, (**9·66**) or (**9·662**), obtained by assigning to M every positive and negative value, are separated by the asymptotic cone

$$S_{xx}x^2 + S_{yy}y^2 + S_{zz}z^2 = 0 \quad . \quad . \quad . \quad (9·67)$$

for all radial directions along which the elongation is zero. This is a special case of the cone of constant elongation for which

$$\frac{M}{\mathbf{r}^2} = K,$$

where K is the constant elongation. Substituting in (**9·662**) we find for the equation of this cone, since $\mathbf{r}^2 = x^2 + y^2 + z^2$,

$$(S_{xx} - K)x^2 + (S_{yy} - K)y^2 + (S_{zz} - K)z^2 = 0 \ . \ (9·671)$$

When the principal elongations are all equal we have a **uniform dilatation.** It will be remembered that the terms dilatation, elongation, etc., are used in an algebraical way ; for instance a uniform contraction will be treated as a dilatation by using a negative sign. Another simple type of strain is the **simple shear,** for which one of the principal elongations is zero, while the remaining two are numerically equal ; but have opposite signs. For example $\dfrac{\partial \gamma}{\partial z} = 0$ while $\dfrac{\partial \alpha}{\partial x} = -\dfrac{\partial \beta}{\partial y}$. It will be noticed that there is no change in volume since $\mathbf{div}\ (\alpha, \beta, \gamma)$ is zero. We shall use the term **shear** for any pure strain not associated with a change in volume. Any pure strain may be regarded as a superposition on one another of a uniform dilatation and simple shears ; for any pure strain consists in three elonga-tions $\dfrac{\partial \alpha}{\partial x}$, $\dfrac{\partial \beta}{\partial y}$ and $\dfrac{\partial \gamma}{\partial z}$ in the directions of its principal axes and

$$\frac{\partial \alpha}{\partial x} = \tfrac{1}{3}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}\right) + \tfrac{1}{3}\left(\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y}\right) + \tfrac{1}{3}\left(\frac{\partial \alpha}{\partial x} - \frac{\partial \gamma}{\partial z}\right),$$

$$\frac{\partial \beta}{\partial y} = \tfrac{1}{3}\left(\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x}\right) + \tfrac{1}{3}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}\right) + \tfrac{1}{3}\left(\frac{\partial \beta}{\partial y} - \frac{\partial \gamma}{\partial z}\right),$$

$$\frac{\partial \gamma}{\partial z} = \tfrac{1}{3}\left(\frac{\partial \gamma}{\partial z} - \frac{\partial \alpha}{\partial x}\right) + \tfrac{1}{3}\left(\frac{\partial \gamma}{\partial z} - \frac{\partial \beta}{\partial y}\right) + \tfrac{1}{3}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}\right) \quad . \quad (9·68)$$

The strain therefore consists of a uniform dilatation which may be regarded as due to three principal elongations each equal to $\tfrac{1}{3}\left(\dfrac{\partial \alpha}{\partial x} + \dfrac{\partial \beta}{\partial y} + \dfrac{\partial \gamma}{\partial z}\right)$; a simple shear associated with the X and Y axes consisting in an X elongation of $\tfrac{1}{3}\left(\dfrac{\partial \alpha}{\partial x} - \dfrac{\partial \beta}{\partial y}\right)$ and a Y

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elongation of $\frac{1}{3}\left(\dfrac{\partial \beta}{\partial y} - \dfrac{\partial \alpha}{\partial x}\right)$ and two other simple shears associated with the YZ and ZX pairs of axes respectively. It follows too that any shear can be regarded as a superposition of a number of simple shears.

Let $a\,b\,c\,d$, Fig. 9·61, represent a cubical portion of the medium, each side of which is taken, for convenience, to be 2 cm. in length, and let it be given numerically equal elongations in the X and Y directions, the former positive and the

Y

O

X
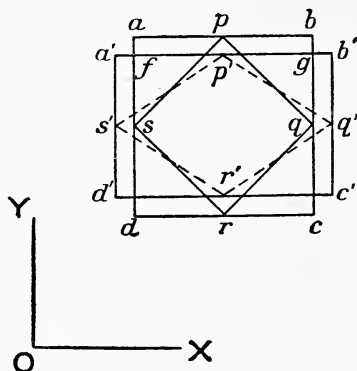
Fig. 9·61

latter negative. Since elongation means increase in length per unit length, the block will be stretched so that

$$\frac{\partial \alpha}{\partial x} = (a'f) = (gb') = e, \text{ say,}$$

while
$$-\frac{\partial \beta}{\partial y} = (af) = (bg) = e.$$

Its dimensions in the Z direction are unaffected. Consider the portion of the cube cut out by four planes parallel to Z and bisecting ab, bc, cd and da along the lines p, q, r and s. This portion of the block becomes, on shearing, p', q', r', s'. It is easy to see that the small angle, ε, between pq and $p'q'$ is equal to the elongation, e. It is in fact equal to $\dfrac{(qq')}{\sqrt{2}}$ divided by one half of pq or

$$\frac{(qq')}{\sqrt{2}} \Big/ \tfrac{1}{2}\sqrt{2}.$$

If the sheared block be turned, so as to bring the face $r's'$ into

coincidence with rs, the faces $p's$ and $q'r$ will be inclined to ps and qr by an amount

$$\phi = 2\varepsilon = 2e, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9\cdot69)$$

(see Fig. 9·62). The angle $\phi = 2e$ is usually taken as a measure of the simple shear. If the sides, ps and sr, of the unsheared
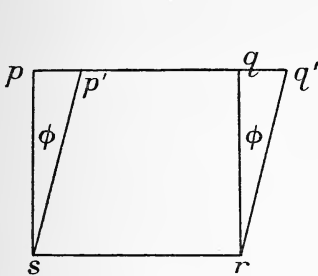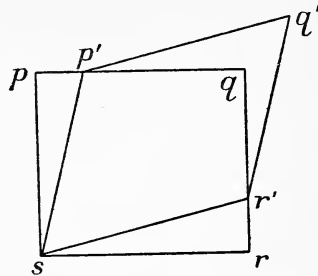
FIG. 9·62　　　　　　　　FIG. 9·63

block are parallel to the co-ordinate axes, it is evident that

$$\phi = \angle psp' + \angle rsr' \quad \text{(Fig. 9·63)},$$

or

$$\phi = \frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad (9\cdot691)$$

The physical meanings of all the components of the strain tensor are now evident.

§ 9·7.　STRESS

A condition of strain may be set up in a medium in various ways ; for example by gravity or by electric and magnetic fields. Every material medium is normally slightly strained by reason of its weight. The insulating medium, glass, mica or ebonite,
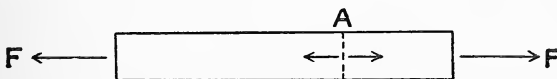
FIG. 9·7

between the plates of a condenser is in a state of strain when the condenser is charged. Weight and electric or magnetic forces are examples of impressed forces which bring about a condition of strain in material media. Correlated with the **strain** at any point in a medium we have a corresponding state of **stress**, which is evoked (in accordance with Newton's third law) by the impressed forces producing the strain. To fix our ideas, suppose a cylindrical rod (Fig. 9·7) to be strained by

numerically equal forces, F, applied at its ends and stretching it along its axis. It is evident that the material to the left of any cross-section, A, will experience a force, F, directed to the right while a numerically equal force, in the opposite sense, will be exerted on the material to the right of the section. The term **stress** in its widest sense is applied to forces of this kind. It is clear that, in order to specify completely the state of stress at any point in a material medium, we must be able to express the magnitude and direction of the force per unit area on any small area in the neighbourhood of the point, for any orientation of this area.

Consider a small element of area, $\mathbf{dS}$, (Fig. 9·71) in a continuous medium. It will be helpful to follow our usual practice and regard it as a vector. We shall imagine an arrow drawn perpendicular to $\mathbf{dS}$ and having a length numerically equal to it. The components of $\mathbf{dS}$, namely dS_x, dS_y and dS_z, will be equal to the projections of $\mathbf{dS}$ on the YZ, ZX and XY planes respectively, provided these are furnished with appropriate signs. If $\mathbf{f}$ be the force exerted by the medium, situated on the side of $\mathbf{dS}$ to which the arrow is directed, on that situated on the other side, we may express its X component in the form
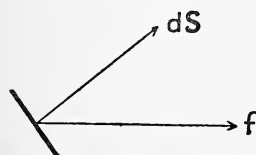
FIG. 9·71

$$f_x = t_{xn}\mathbf{dS} \quad . \quad . \quad . \quad . \quad . \quad (9\cdot7)$$

so that t_{xn} is the X component of the force on $\mathbf{dS}$, reckoned per unit area.[1] Sometimes it will be convenient to use the alternative definition,

$$f_x = -p_{xn}\mathbf{dS} \quad . \quad . \quad . \quad . \quad (9\cdot701)$$

or
$$f_x{}' = p_{xn}dS,$$

in which $\mathbf{f}' = -\mathbf{f}$ is the force exerted on the medium *on the same side* of $\mathbf{dS}$ as that to which the arrow (Fig. 9·71) is directed. By definition, therefore,

$$p_{xn} = -t_{xn} \quad . \quad . \quad . \quad . \quad . \quad (9\cdot702)$$

The component, f_x, can be expressed as the sum of three terms, in the following way : Let $\mathbf{dS}$ be the face, *abc*, of a tetrahedron *oabc* (Fig. 9·72). The components of $\mathbf{dS}$ are dS_x in the direction X, equal to the area *obc* ; and dS_y in the direction Y, equal to the area *oca* ; and dS_z in the direction Z, equal to the

---

[1] The plan is adopted here of using the first subscript, in this case x, to indicate the component of the force, and the second subscript, n, to indicate the direction of the vector $\mathbf{dS}$.

area *oab*. The X component of the force on the face *obc* of the tetrahedron will be denoted by

$$p_{xx}dS_x \text{ or } -t_{xx}dS_x$$

in accordance with the definitions and notation in (**9·701**) and (**9·702**). Similarly the X components of the forces on the faces *oca* and *oab* of the tetrahedron will be

$$- t_{xy}dS_y,$$
and
$$- t_{xz}dS_z$$

respectively. Therefore the total value of the X component of the force, *due to stress*, on the tetrahedron is

$$f_x - (t_{xx}dS_x + t_{xy}dS_y + t_{xz}dS_z) \quad . \quad . \quad . \quad (9\cdot71)$$

To this we have to add a force equal to the volume of the tetrahedron multiplied by R_x, the X component of **R**, the impressed force, or so-called body force, reckoned per unit volume. This is a force of external origin, due to gravitation or other causes, and it will become negligible in comparison with the forces over the surface of the tetrahedron as the dimensions of the latter approach the limit zero. This becomes evident when we reflect that dividing the lengths of the edges of the tetrahedron by n
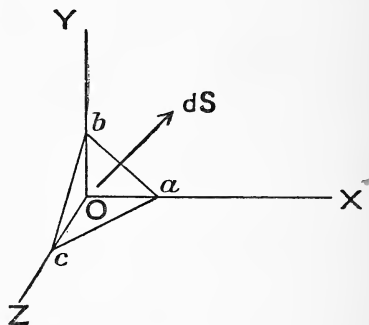
FIG. 9·72

reduces the area of any face to $\dfrac{1}{n^2}$ of its original area, while

the volume becomes $\dfrac{1}{n^3}$ of the original volume. The expression,

(**9·71**), therefore represents in the limit the X component of the resultant force on the tetrahedron. It must therefore be equal to the mass of the tetrahedron multiplied by the X component of its acceleration. But for finite accelerations this product must also be negligible for the same reason which led us to neglect the body force. Consequently we have to equate the expression (**9·71**) to zero, and remembering that we have similar equations associated with the Y and Z axes, we arrive at the result

$$f_x = t_{xn}\mathbf{dS} = t_{xx}dS_x + t_{xy}dS_y + t_{xz}dS_z,$$
$$f_y = t_{yn}\mathbf{dS} = t_{yx}dS_x + t_{yy}dS_y + t_{yz}dS_z,$$
$$f_z = t_{zn}\mathbf{dS} = t_{zx}dS_x + t_{zy}dS_y + t_{zz}dS_z. \quad . \quad . \quad (9\cdot72)$$

The quantities

$$t_{xx}, \ t_{xy}, \ t_{xz},$$
$$t_{yx}, \ t_{yy}, \ t_{yz}, \qquad \cdot$$
$$t_{zx}, \ t_{zy}, \ t_{zz}, \qquad \cdots \qquad \textbf{(9·721)}$$

constitute a tensor of the second rank, as the form of the equations (**9·72**) suggests. We shall refer to it as the **stress tensor.** The component t_{xy}, for example, means the X component of the force per unit area on a (small) surface perpendicular to the Y axis (i.e. its vector arrow is in the direction of the Y axis), and, further, it is the force exerted on the medium situated on the side a (Fig. 9·73) by the medium situated on the side b.
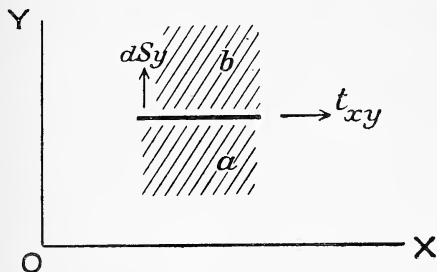
Fig. 9·73

The tensor character of (**9·721**) can be demonstrated in a simple way. Let us write the first equation (**9·72**) in the form

$$f_x = t_{xx}S_x + t_{xy}S_y + t_{xz}S_z,$$

where S_x, S_y and S_z are of course small, and are the components of a vector **S**. Therefore in any small neighbourhood f_x is a (linear) function of S_x, S_y and S_z,

$$f_x = f_x(S_x, \ S_y, \ S_z),$$

and

$$\frac{\partial f_x}{\partial S_y} = t_{xy}.$$

Now, if S_y is the Y component of a vector, the operation $\dfrac{\partial}{\partial S_y}$ transforms according to the rule for the Y component of a vector (see equation (**2·41**), and as f_x is the X component of a vector, it follows that $\dfrac{\partial f_x}{\partial S_y}$ transforms according to the same rule as the product, $a_x b_y$, where **a** and **b** are two vectors. Thus $\dfrac{\partial f_x}{\partial S_y}$, or t_{xy}, is the XY component of a tensor of the second rank, according to the definition of § 2·3.

§ 9·8. Stress Quadric. Analysis of Stresses

Imagine a vector, $\mathbf{r} \equiv (x, \ y, \ z)$, parallel to the vector **dS**. We shall think of it as a line drawn in the direction of the arrow

associated with **dS**. Let us further suppose the origin of co-ordinates to be situated in **dS**. We have then

$$\frac{dS_x}{d\mathbf{S}} = \frac{x}{\mathbf{r}} \qquad \ldots \ldots \quad (9\cdot8)$$

and two similar equations associated with the Y and Z axes respectively. We now form the scalar product, $(\mathbf{fr})$, using the equations $(9\cdot72)$.

$$f_x x + f_y y + f_z z = t_{xx} x dS_x + t_{xy} x dS_y + t_{xz} x dS_z$$
$$+ t_{yx} y dS_x + t_{yy} y dS_y + t_{yz} y dS_z$$
$$+ t_{zx} z dS_x + t_{zy} z dS_y + t_{zz} z dS_z.$$

In this equation let us replace the left-hand member by $f_n \mathbf{r}$, where f_n is the component of **f** normal to the surface **dS**, i.e. its component in the direction of the vector **dS** or the vector **r**. On the right-hand side of the equation we replace dS_x, dS_y and dS_z by $\frac{x}{\mathbf{r}}d\mathbf{S}$, $\frac{y}{\mathbf{r}}d\mathbf{S}$ and $\frac{z}{\mathbf{r}}d\mathbf{S}$ respectively (equations $9\cdot8$). In this way we get

$$f_n \mathbf{r}^2 = \{ t_{xx} x^2 + t_{xy} xy + t_{xz} xz$$
$$+ t_{yx} yx + t_{yy} y^2 + t_{yz} yz$$
$$+ t_{zx} zx + t_{zy} zy + t_{zz} z^2 \} d\mathbf{S}.$$

Therefore if t_{nn} is the tension normal to **dS**, i.e. $f_n/d\mathbf{S}$, we have

$$t_{nn} \mathbf{r}^2 = t_{xx} x^2 + t_{xy} xy + t_{xz} z^2$$
$$+ t_{yx} yx + t_{yy} y^2 + t_{yz} yz$$
$$+ t_{zx} zx + t_{zy} zy + t_{zz} z^2. \qquad \ldots \quad (9\cdot81)$$

In this equation, t_{xx}, t_{xy}, etc., are the components of the stress at the origin. They are therefore constants, i.e. not functions of x, y and z in the equation $(9\cdot81)$. If now we replace $t_{nn} \mathbf{r}^2$ by a constant, M, which may conveniently have the numerical, or absolute value 1, we obtain

$$t_{xx} x^2 + t_{xy} xy + t_{xz} xz$$
$$+ t_{yx} yx + t_{yy} y^2 + t_{yz} yz$$
$$+ t_{zx} zx + t_{zy} zy + t_{zz} z^2 = M \qquad \ldots \quad (9\cdot82)$$

which is the equation of a quadric surface. It is called the **stress quadric**. Obviously a suitable rotation of the co-ordinate axes reduces $(9\cdot82)$ to the simpler form

$$T_{xx} x^2 + T_{yy} y^2 + T_{zz} z^2 = M \qquad \ldots \quad (9\cdot821)$$

The new co-ordinate axes are naturally termed the **principal axes** of the stress, and T_{xx}, T_{yy} and T_{zz}, the values which t_{xx}, t_{yy} and t_{zz} assume for these special co-ordinates, may be called the **principal tensions** or **stresses** (T_{xy}, T_{xz}, T_{yz}, etc., are of course zero). The tension t_{nn} or T_{nn} (normal to the surface

element **dS**) is equal to $M/\mathbf{r}^2$ and therefore the quadric has the property, that the normal tension in any direction is inversely proportional to the square of the radius vector of the quadric in that direction ; and this applies also to the normal pressure p_{nn} or $P_{nn} \equiv (-t_{nn}, -T_{nn})$.   All that has been said about the relationship between the strain quadric and the radial elongation applies, *mutatis mutandis*, to the stress quadric and the normal tension.   If the quadric is an hyperboloid, there will be a conjugate hyperboloid, obtained by changing the sign of M, and an asymptotic cone, analogous to (**9·67**), separating them and representing the directions along which the normal tension (or pressure) vanishes.   There will also be a cone of constant normal tension analogous to (**9·671**).

It appears then that any state of stress can be regarded as due to three principal tensions, T_{xx}, T_{yy}, T_{zz} (or pressures P_{xx}, P_{yy}, P_{zz}) in directions perpendicular to one another.   When the principal stresses are equal to one another ($T_{xx} = T_{yy} = T_{zz}$) we have a uniform traction (dilating stress) or a uniform pressure. In an isotropic medium this must give rise to a uniform dilatation (or compression).   A tension T_{xx}, normal to the YZ plane together with a numerically equal one of opposite sign normal to the ZX plane, $T_{yy} = -T_{xx}$, we shall term a simple shearing stress, since it will produce a simple shear in an isotropic medium. Obviously the two tensions produce numerically equal elongations of opposite sign normal to the YZ and ZX planes while the elongations which they would separately produce normal to the XY plane will also be numerically equal and of opposite signs, so that the resulting elongation normal to the XY plane is zero.

Since any state of stress can be regarded as three tensions (positive or negative) in mutually perpendicular directions we may look upon it as a superposition of simple shearing stresses on a uniformly dilating stress.   In fact

$$T_{xx} = \tfrac{1}{3}(T_{xx} + T_{yy} + T_{zz}) + \tfrac{1}{3}(T_{xx} - T_{yy}) + \tfrac{1}{3}(T_{xx} - T_{zz})$$
$$T_{yy} = \tfrac{1}{3}(T_{yy} - T_{xx}) + \tfrac{1}{3}(T_{xx} + T_{yy} + T_{zz}) + \tfrac{1}{3}(T_{yy} - T_{zz}),$$
$$T_{zz} = \tfrac{1}{3}(T_{zz} - T_{xx}) + \tfrac{1}{3}(T_{zz} - T_{yy}) + \tfrac{1}{3}(T_{xx} + T_{yy} + T_{zz}) \quad \textbf{(9·83)}$$

We have here a complete analogy with a homogeneous pure strain (equation **9·68**).

There is an alternative way of describing a simple shearing stress.   To show this let us consider an element of the medium in the form of a prism and having its axis parallel to the Z axis. We shall suppose its cross-section to be an equilateral rightangled triangle (*aob*, Fig. 9·8), the sides *oa* and *ob* being perpendicular to the X and Y axes respectively, and each equal in area to unity.   The shearing stress may be a force T, over the side

oa, parallel to the X axis and an equal force T over the side *ob* and in a direction opposite to that of the Y axis. Since body forces may be ignored for the reason already explained, these two forces will produce a resultant tangential force over the side *ba* of the prism and equal to $\sqrt{2}\,|\ T$. But the area of *ab* is $\sqrt{2}\,|$. Therefore the tangential stress is equal to T. We conclude therefore that we may describe a simple shearing stress as made up of two numerically equal normal stresses perpendicular to one another or, alternatively, as consisting of a single tangential stress at 45 degrees to the normal stresses.
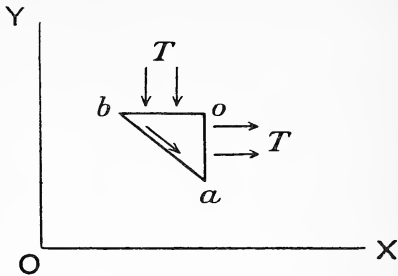
Fig. 9·8

§ 9·9. Force and Stress

We shall next consider the resultant force exerted on the
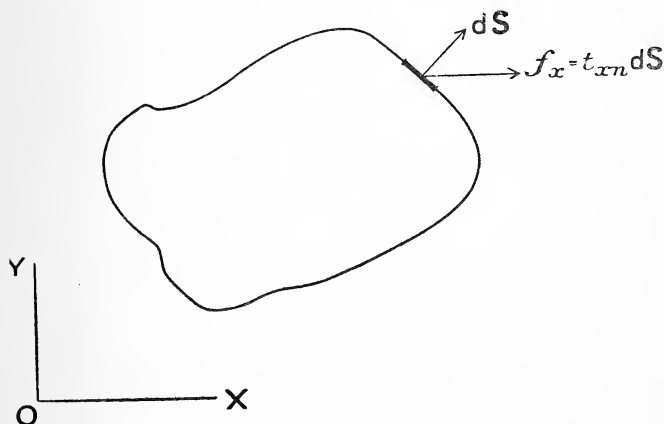
Fig. 9·9

medium within a closed surface in consequence of a state of stress. Its X component is (see Fig. 9·9)

$$F_x = \Sigma f_x = \iint t_{xn}\mathrm{d}\mathbf{S},$$

or

$$F_x = \iint \left\{ t_{xx}dS_x + t_{xy}dS_y + t_{xz}dS_z \right\}$$

(§ 9·7). This is equivalent (§ 3·2) to

$$F_x = \int \int \int \left\{ \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} \right\} dx \, dy \, dz \qquad . \quad . \; \textbf{(9·9)}$$

where the integration is extended over the whole of the volume enclosed by the surface. Since equation (9·9) must be valid however small the enclosed volume may be, the X component of the force exerted on a volume element $dx \, dy \, dz$ must be equal to

$$\left\{ \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} \right\} dx \, dy \, dz \quad . \quad . \quad . \; \textbf{(9·901)}$$

and consequently the X component of the force per unit volume at any point must be equal to

$$\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} \quad . \quad . \quad . \quad . \; \textbf{(9·91)}$$

For the Y and Z components we find respectively

$$\frac{\partial t_{yx}}{\partial x} + \frac{\partial t_{yy}}{\partial y} + \frac{\partial t_{yz}}{\partial z}$$

$$\frac{\partial t_{zx}}{\partial x} + \frac{\partial t_{zy}}{\partial y} + \frac{\partial t_{zz}}{\partial z} \quad . \quad . \quad . \quad . \; \textbf{(9·91)}$$

For brevity these expressions, which are divergences according to the extended modern use of the term, may be written as

$$(\textbf{div t})_x$$
$$(\textbf{div t})_y$$
$$(\textbf{div t})_z \quad . \quad . \quad . \quad . \quad . \; \textbf{(9·911)}$$

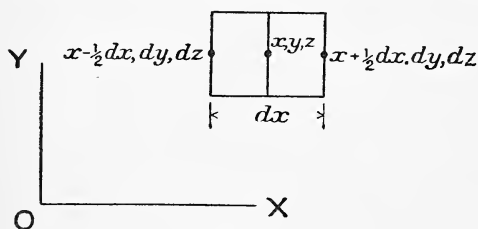Incidentally it may be remarked that the divergence of a vector (tensor of rank 1) is a scalar quantity (tensor of rank 0); the divergence of a tensor of rank 2 (the present instance) is a vector (tensor of rank 1) and quite generally the divergence of a tensor of rank n is a tensor of rank $n - 1$.

The result expressed by equations (9·91) is so important that it is worth while to arrive at

FIG. 9·91

it directly, without employing the theorem of Gauss. Let (x, y, z) be the co-ordinates of the central point of a volume element $dx \, dy \, dz$ of the medium, and imagine a plane surface perpendicular to the X axis and bisecting the element, Fig. 9·91. The X com-

ponent of the force exerted over this plane on the part of the element to the left of it is

$$\bar{t}_{xx}\, dy\, dz,$$

$\bar{t}_{xx}$ meaning the average value of t_{xx} over the plane in question. Therefore the X component of the force on the face $dy\, dz$ on the right of the element must be

$$\left\{\bar{t}_{xx} + \tfrac{1}{2}\frac{\partial \bar{t}_{xx}}{\partial x}dx\right\} dy\, dz.$$

This is a force tending to drag the element in the X direction. In the same way it will be seen that a force

$$\left\{\bar{t}_{xx} - \tfrac{1}{2}\frac{\partial \bar{t}_{xx}}{\partial x}dx\right\} dy\, dz,$$

tending to drag the element to the left, is exerted over the face $dy\, dz$ on the left. Consequently the resulting X component of the force on the volume element, so far as it is due to stresses on the faces perpendicular to the X axis, will be

$$\frac{\partial \bar{t}_{xx}}{\partial x}dx\, dy\, dz,$$

which reduces to

$$\frac{\partial t_{xx}}{\partial x}dx\, dy\, dz$$

in the limit when dx, dy and dz are sufficiently small. In a similar way we may show that the part of the X component of the force exerted on the element, in consequence of the stresses over the faces perpendicular to the Y axis, is

$$\frac{\partial t_{xy}}{\partial y}dx\, dy\, dz,$$

while that due to stresses over the faces perpendicular to the Z axis is

$$\frac{\partial t_{xz}}{\partial z}dx\, dy\, dz.$$

On adding all three together we arrive at the expression we found by the use of the theorem of Gauss. We may of course replace t_{xx}, t_{xy} and t_{xz} by $-p_{xx}$, $-p_{xy}$ and $-p_{xz}$ respectively (9·702) and thus obtain the alternative expression,

$$-\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z}\right) \quad . \quad . \quad . \quad (9·912)$$

for the force per unit volume.

The x, y and z in the foregoing equations (**9·91**), etc., refer to the *actual* or instantaneous positions of the parts of the medium and not to their positions in its undeformed state. We ought therefore to have used the letters ξ, η and ζ in order to avoid confusion and possible error. If, however, as we are assuming, the differential quotients $\dfrac{\partial \alpha}{\partial x}$, $\dfrac{\partial \beta}{\partial y}$, etc., in the strain tensor are negligible by comparison with unity, no errors will arise if we use x, y and z in sense defined in the description of strain. To show that this is the case, consider the differential quotient $\dfrac{\partial \phi}{\partial x}$, where ϕ may mean a stress t_{xx}, or any other function of (x, y, z) or (ξ, η, ζ). Since

$$\xi = x + \alpha,$$
$$\eta = y + \beta,$$
$$\zeta = z + \gamma,$$

(equations **9·51**),

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial \zeta}\frac{\partial \zeta}{\partial x},$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \xi}\left(1 + \frac{\partial \alpha}{\partial x}\right) + \frac{\partial \phi}{\partial \eta}\frac{\partial \beta}{\partial x} + \frac{\partial \phi}{\partial \zeta}\frac{\partial \gamma}{\partial x},$$

or

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \xi} + \frac{\partial \phi}{\partial \xi}\frac{\partial \alpha}{\partial x} + \frac{\partial \phi}{\partial \eta}\frac{\partial \beta}{\partial x} + \frac{\partial \phi}{\partial \zeta}\frac{\partial \gamma}{\partial x};$$

and this reduces to

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \xi}$$

when $\dfrac{\partial \alpha}{\partial x}$, $\dfrac{\partial \beta}{\partial x}$, etc., are very small compared with unity.

§ 10. Hooke's Law—Moduli of Elasticity

The question now arises : What is the relationship between a state of strain in a medium and the correlated stress ? Generally speaking the relationships between physical quantities can be expressed by *analytic functions*. It is probable that this statement is *strictly* true when it is confined to the quantitative relationships in *macroscopic* phenomena. The phenomena of elasticity, with which we are now concerned, come under this heading. In fact in § 9·5 we assumed that even the volume element $dx\,dy\,dz$ was very large when measured by the scale of the granular structure of the medium. Roughly speaking, an analytic function is one which can be expanded by Taylor's

theorem. If θ, ϕ and ψ are the independent variables in such a function, any sufficiently small increments $d\theta$, $d\phi$ and $d\psi$ will give rise to an increment of the function equal to

$$df = \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi + \frac{\partial f}{\partial \psi}d\psi,$$

the differential quotients $\frac{\partial f}{\partial \theta}$, etc., being independent of the increments $d\theta$, $d\phi$, $d\psi$. We should therefore expect *a priori* that the components of the stress tensor are linear functions of those of the strain tensor when these latter are small. Experiment shows that this is the case. We have here in fact a slight generalization of the law stated by Robert Hooke (1635–1703) in the famous anagram *ce iii n o sss tt uu*($\equiv$ *ut Tensio sic Vis*).

First of all let us consider a uniform dilatation. Of the components of the strain tensor all vanish except $\frac{\partial \alpha}{\partial x}$, $\frac{\partial \beta}{\partial y}$, $\frac{\partial \gamma}{\partial z}$ and

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y} = \frac{\partial \gamma}{\partial z} = \tfrac{1}{3} \times \textbf{dilatation}.$$

In an *isotropic* medium therefore

$$t_{xx} = t_{yy} = t_{zz}$$

and Hooke's law requires

$$t_{xx} = k \times \textbf{dilatation},$$

where k is a constant, called the **bulk modulus of elasticity**. Therefore

$$t_{xx} = 3k \times \frac{\partial \alpha}{\partial x} \qquad \text{.   .   .   .   .   .} \quad \textbf{(10)}$$

when the strain is a uniform dilatation. We might of course have defined this modulus as equal to $k' = 3k$ in the equation

$$t_{xx} = k'\frac{\partial \alpha}{\partial x}.$$

The definition given is the one which is universally adopted and is probably the more convenient of the two.

A simple shear may be regarded as due to a tangential stress (§ 9·8). Let us suppose it to be in the XY plane ; then, in accordance with Hooke's law, we have for an isotropic medium

$$t_{xy} = n\phi,$$

or $\qquad\qquad\qquad t_{xy} = n\Big(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}\Big) \qquad \text{.   .   .   .} \quad \textbf{(10·01)}$

where ϕ is the angle of shear (equations **9·69** and **9·691**) and n is a constant called the **simple rigidity** or **modulus of rigidity**.

From (**10·01**) we can derive another equation involving n. We have seen (§ 9·6) that a simple shear is equivalent to two numerically equal elongations, of opposite signs, along lines at right angles to one another and that $\phi = 2e$, i.e. twice the positive elongation. Furthermore instead of attributing the shear to a tangential stress, for example t_{xy} in (**10·01**), we may attribute it to an equal normal stress, t_{xx}, perpendicular to the YZ plane, and a stress, $t_{yy} = -t_{xy}$, perpendicular to ZX plane. Therefore (**10·01**) is equivalent to

$$t_{xx} = 2ne \qquad . \quad . \quad . \quad . \qquad (\mathbf{10\cdot011})$$

or

$$t_{xx} = 2n\frac{\partial \alpha}{\partial x}.$$

From the theoretical point of view these are the simplest relations between stresses and strains. It should be observed that since the effect of a tangential stress is merely to produce a simple shear, equation (**10·01**) is a general expression for t_{xy}; on the other hand equations (**10**) and (**10·011**) are expressions for t_{xx} which are true in special cases only, the former for a uniform dilatation, the latter for the case of a simple shear. We have to search, therefore, for general expressions for t_{xx}, t_{yy} and t_{zz}. The expressions (**9·68**) show the general strain to consist of (a), a uniform dilatation in which each axial elongation is

$$\tfrac{1}{3}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}\right),$$

(b) three simple shears, a typical one consisting of the elongation

$$e = \tfrac{1}{3}\left(\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y}\right), \ \perp \text{ to } YZ,$$

and

$$-e = \tfrac{1}{3}\left(\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x}\right), \ \perp \text{ to } ZX.$$

The dilatation (a), contributes to t_{xx} an amount equal to $3k \times$ (axial elongation) in accordance with (**10**) or

$$3k \times \tfrac{1}{3}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}\right);$$

and under (b), we have a contribution to t_{xx} equal to

$$2n \times \tfrac{1}{3}\left(\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y}\right),$$

and another equal to

$$2n \times \tfrac{1}{3}\left(\frac{\partial \alpha}{\partial x} - \frac{\partial \gamma}{\partial z}\right),$$

in consequence of (**10·011**). Adding all three contributions to t_{xx} we get

$$t_{xx} = k\left(\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial y} + \frac{\partial\gamma}{\partial z}\right) + \frac{2n}{3}\left(\frac{\partial\alpha}{\partial x} - \frac{\partial\beta}{\partial y}\right) + \frac{2n}{3}\left(\frac{\partial\alpha}{\partial x} - \frac{\partial\gamma}{\partial z}\right).$$

This is the general expression for t_{xx}. We may write it and the corresponding expressions for t_{yy} and t_{zz} in the following more compact form :

$$t_{xx} = \left(k + \frac{4n}{3}\right)\frac{\partial\alpha}{\partial x} + \left(k - \frac{2n}{3}\right)\frac{\partial\beta}{\partial y} + \left(k - \frac{2n}{3}\right)\frac{\partial\gamma}{\partial z},$$

$$t_{yy} = \left(k - \frac{2n}{3}\right)\frac{\partial\alpha}{\partial x} + \left(k + \frac{4n}{3}\right)\frac{\partial\beta}{\partial y} + \left(k - \frac{2n}{3}\right)\frac{\partial\gamma}{\partial z},$$

$$t_{zz} = \left(k - \frac{2n}{3}\right)\frac{\partial\alpha}{\partial x} + \left(k - \frac{2n}{3}\right)\frac{\partial\beta}{\partial y} + \left(k + \frac{4n}{3}\right)\frac{\partial\gamma}{\partial z} \quad (\mathbf{10·02})$$

When the state of stress consists of

$$t_{xx} \neq 0,$$
$$t_{yy} = t_{zz} = 0,$$

the elongations $\dfrac{\partial\beta}{\partial y}$ and $\dfrac{\partial\gamma}{\partial z}$ become equal to one another of course, and equations (**10·02**) become

$$t_{xx} = \left(k + \frac{4n}{3}\right)\frac{\partial\alpha}{\partial x} + 2\left(k - \frac{2n}{3}\right)\frac{\partial\beta}{\partial y},$$

$$0 = \left(k - \frac{2n}{3}\right)\frac{\partial\alpha}{\partial x} + 2\left(k + \frac{n}{3}\right)\frac{\partial\beta}{\partial y}.$$

Eliminating $\dfrac{\partial\beta}{\partial y}$ we find

$$t_{xx} = \left(\frac{9nk}{3k+n}\right)\frac{\partial\alpha}{\partial x} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (\mathbf{10·03})$$

and for ratio, $s = -\dfrac{\partial\beta}{\partial y}\Big/\dfrac{\partial\alpha}{\partial x}$,

$$s = \frac{3k - 2n}{2(3k+n)} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (\mathbf{10·04})$$

The constant

$$Y = \frac{9nk}{3k+n} \quad \cdot \quad \cdot \quad \cdot \quad (\mathbf{10·041})$$

is called **Young's modulus** of elasticity, and the ratio, s, of the lateral contraction to the longitudinal elongation is known as **Poisson's ratio.** Young's modulus, Y, and the modulus of rigidity, n, can easily be determined experimentally and the formula (**10·041**) enables us to find the bulk modulus, k, from the experimentally determined values of Y and n.

§ 10·1. Thermal Conditions. Elastic Moduli of Liquids and Gases

It is convenient to speak of a body or a medium as *elastic* when there is a linear relationship between stress and strain or between a small change in the stress and the resulting deformation. The analysis in the foregoing paragraphs tacitly assumes that an elastic body, or a portion of an elastic medium, has a finite and determinate volume even when the stress components are all zero. It is thus restricted to solid and liquid media, the latter being media for which $n = 0$ and in which there are consequently no shearing stresses (see equation **10·01**) of the elastic type. In a liquid therefore the stresses are all normal stresses. It is quite true that in actual liquids and gases we may have shearing stresses, due to viscosity; but we are confining our attention at this stage to cases where such stresses may be ignored.

If n is made equal to zero in (**10·02**) it will be seen that the state of stress in a liquid is a uniform dilating (or compressing) stress and

$$t_{xx} = t_{yy} = t_{zz} = k\Big(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}\Big),$$

or, writing t for t_{xx}, we have

$$t = k\delta V/V,$$

or

$$p = -k\delta V/V \quad . \quad . \quad . \quad . \quad (10·1)$$

In these equations t is the tension at the point in question, $p = -t$ is the pressure and the divergence has been replaced by its equivalent $\delta V/V$ or the increment in volume per unit volume. There is clearly only one modulus in the case of a liquid, namely the bulk modulus, k. It should be noted that t may be *positive* as well as negative in the case of a liquid. That is to say it is possible to develop in a liquid a condition of stress giving rise to a positive dilatation. If a glass vessel with fairly strong walls and a narrow stem (after the fashion of an ordinary mercury thermometer) be nearly filled with water from which air and dissolved gases have been expelled by prolonged boiling, and if it be sealed off while the water is boiling in the upper part of the stem, so as to enclose nothing but water and water vapour, we have a state of affairs in which the closed vessel is full of (liquid) water except for a very small space at the top of the stem which contains only water vapour. By judiciously warming the vessel and contained water the latter may be caused to expand till it fills the whole vessel and presses hard against its walls without however developing a pressure big enough to

break the vessel. If now it be allowed to cool the liquid is still firmly held to the sides of the vessel and continues to fill it ; but it is now in a state of tension. Gases differ from liquids in that a state of *positive* tension cannot be produced in them. In fact a gas is always subject to a positive pressure (negative tension) which can only approach the limit zero when the volume of the gas becomes very great. Its elastic behaviour can how-ever be brought within the scope of the preceding theory if we agree to use the term ' stress ' for any *small change* in the pressure of the gas. For gases therefore equation (**10·1**) becomes

$$\delta p = -k\frac{\delta V}{V} \quad . \quad . \quad . \quad (\mathbf{10\cdot101})$$

In § 10 it is implied that stress and strain mutually determine one another ; that for instance the components of the strain tensor are uniquely determined by those of the stress tensor and *vice versa*. Now small changes in temperature can bring about appreciable volume changes while the condition of stress is maintained constant. Such volume changes are relatively enormous in the case of gases. It is therefore important that definite thermal conditions should be laid down in dealing with elastic phenomena. Unless the contrary is stated or clearly implied we shall take the temperature to be constant without expressly mentioning this condition. That is to say we shall suppose the strain to occur under *isothermal* conditions. There is however one other thermal condition, or set of conditions, in which we are specially interested and which may be called *adiabatic* or *isentropic*. We shall understand by an adiabatic strain one which is produced very slowly and in such a way that heat is prevented from entering or leaving the strained medium. The isothermal relation between the pressure and volume of a given mass of gas is approximately expressed by

$$pv = \text{constant (Boyle's law)},$$

and therefore $\qquad \delta p = -p\dfrac{\delta v}{v},$

so that under isothermal conditions (equation **10·101**)

$$k = p \quad . \quad . \quad . \quad . \quad . \quad (\mathbf{10\cdot11})$$

Therefore the isothermal bulk modulus of elasticity, or briefly the isothermal elasticity of a gas is equal its pressure.

The adiabatic relation between pressure and volume in the case of a given mass of a gas is approximately

$$pv^\gamma = \text{constant},$$

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where γ is a constant which varies from one gas to another. Consequently we have

$$\delta p = -\gamma p \frac{\delta v}{v}$$

and therefore (equation **10·101**), the adiabatic elasticity of a gas is

$$k = \gamma p \quad . \quad . \quad . \quad . \quad . \quad . \quad \textbf{(10·12)}$$

§ 10·2. DIFFERENTIAL EQUATION OF STRAIN. WAVES IN ELASTIC MEDIA

When we equate the force per unit volume of the medium to the product of its density (mass per unit volume) and its acceleration we have the equation

$$\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} + R_x = \varrho \frac{\partial^2 a}{\partial t^2} \quad . \quad . \quad . \quad \textbf{(10·2)}$$

Here $\mathbf{R} \equiv (R_x, R_y, R_z)$ is the so-called body force per unit volume and ϱ is the density. Substituting for t_{xx} the expression in (**10·02**), for t_{xy} the expression (**10·01**) and the analogous expression for t_{xz} we get

$$\frac{\partial}{\partial x}\left\{\left(k + \frac{4n}{3}\right)\frac{\partial a}{\partial x} + \left(k - \frac{2n}{3}\right)\frac{\partial \beta}{\partial y} + \left(k - \frac{2n}{3}\right)\frac{\partial \gamma}{\partial z}\right\}$$

$$+ \frac{\partial}{\partial y}\left\{n\left(\frac{\partial a}{\partial y} + \frac{\partial \beta}{\partial x}\right)\right\} + \frac{\partial}{\partial z}\left\{n\left(\frac{\partial a}{\partial z} + \frac{\partial \gamma}{\partial x}\right)\right\} + R_x = \varrho \frac{\partial^2 a}{\partial t^2}.$$

After a little reduction this becomes

$$n\left(\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2}\right) + \left(k + \frac{n}{3}\right)\frac{\partial}{\partial x}\left(\frac{\partial a}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}\right) + R_x = \varrho\frac{\partial^2 a}{\partial t^2},$$

or

$$\boxed{n\nabla^2 a + \left(k + \frac{n}{3}\right)\frac{\partial}{\partial x}\{\text{div}\,(a, \beta, \gamma)\} + R_x = \varrho\frac{\partial^2 a}{\partial t^2}} \quad \textbf{(10·201)}$$

and we derive, of course, two similar equations from the Y and Z components of the force per unit volume. If the body force $\mathbf{R}$ is negligible or zero (**10·201**) is essentially identical with the wave equation (**9·23**). Instead of the vector $\boldsymbol{\psi} \equiv (\psi_x, \psi_y, \psi_z)$ in (**9·23**) we have the vector (a, β, γ); instead of the constant A in (**9·23**) we have here the constant n/ϱ and instead of the constant B we now have $\left(k + \frac{n}{3}\right)/\varrho$. The discussion in § 9·2 enables us to infer, therefore, that when a small strain is produced in an elastic solid two waves will travel outwards from

the centre of disturbance, a longitudinal (or dilatational) wave with a velocity

$$v_\lambda = \sqrt{\frac{n + \left(k + \dfrac{n}{3}\right)}{\varrho}} = \sqrt{\frac{k + \dfrac{4n}{3}}{\varrho}} \quad . \quad (10\cdot21)$$

and a transverse (or distortional) wave with a velocity

$$v_\tau = \sqrt{\frac{n}{\varrho}} \quad . \quad . \quad . \quad (10\cdot211)$$

In the cases of liquids and gases, for which $n = 0$, transverse waves obviously cannot be propagated, and the expression for the velocity of longitudinal waves in such media simplifies to

$$v_\lambda = \sqrt{\frac{k}{\varrho}} \quad . \quad . \quad . \quad (10\cdot212)$$

The expressions $(10\cdot21)$ and $(10\cdot211)$ can be verified by considering a plane wave travelling in the X direction. In this case the differential quotients, $\partial/\partial y$ and $\partial/\partial z$, with respect to the Y and Z axes are all zero and equations $(10\cdot201)$ reduce to

$$\left(k + \frac{4n}{3}\right)\frac{\partial^2 a}{\partial x^2} = \varrho\frac{\partial^2 a}{\partial t^2},$$

$$n\frac{\partial^2 \beta}{\partial x^2} = \varrho\frac{\partial^2 \beta}{\partial t^2},$$

$$n\frac{\partial^2 \gamma}{\partial x^2} = \varrho\frac{\partial^2 \gamma}{\partial t^2}.$$

The longitudinal wave was one of the difficulties in the elastic solid theories of light of Fresnel, Neumann and MacCullagh. There are no optical phenomena requiring such a wave. The difficulty was at first imperfectly met by assuming the luminiferous medium to be incompressible, i.e. by assuming $\mathbf{div}\ (a,\ \beta,\ \gamma) = 0$. This assumption makes k infinite, if the stresses are not zero, and hence the longitudinal wave travels with an infinite velocity. While getting rid of the longitudinal wave the assumption, $\mathbf{div}\ (a,\ \beta,\ \gamma) = 0$, led to insurmountable difficulties in other directions. Lord Kelvin solved the difficulty (so far as the wave phenomena of light are concerned ; there are other phenomena which make the hypothesis of an elastic solid aether untenable) by the bold, but not very credible hypothesis that

$$k + \frac{4n}{3} = 0$$

or

$$k = -\frac{4n}{3}.$$

This contractile aether banished the longitudinal wave by making it travel with zero velocity and it was shown by Willard Gibbs and others that it was adequate in other respects.

We might be tempted to adopt the expression (**10·21**) for the velocity of a longitudinal disturbance along a thin rod. Closer investigation however shows that this would be an error. Let AB (Fig. 10·2) represent an element of the rod dx in length parallel to the X axis and suppose x to be the co-ordinate of the
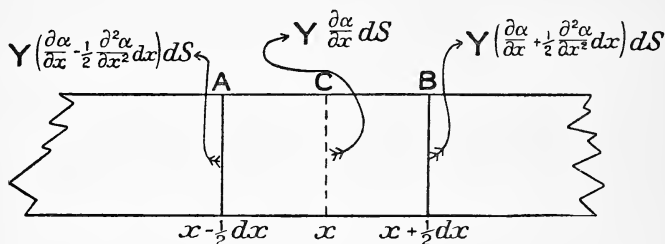

$$Y\left(\frac{\partial\alpha}{\partial x}-\tfrac{1}{2}\frac{\partial^2\alpha}{\partial x^2}dx\right)dS \qquad Y\frac{\partial\alpha}{\partial x}dS \qquad Y\left(\frac{\partial\alpha}{\partial x}+\tfrac{1}{2}\frac{\partial^2\alpha}{\partial x^2}dx\right)dS$$

$$x-\tfrac{1}{2}dx \qquad x \qquad x+\tfrac{1}{2}dx$$

Fig. 10·2

middle point or section of the rod, C. The force exerted over the area dS of the cross-section C must be equal to

$$Y\frac{\partial a}{\partial x}dS,$$

since the tension (force per unit area) is equal to the product of Young's modulus and the elongation. Therefore the force over the section B, tending to pull the element to the right, will be equal to

$$Y\left(\frac{\partial a}{\partial x}+\tfrac{1}{2}\frac{\partial^2 a}{\partial x^2}dx\right)dS.$$

The force exerted over the section A, and pulling the element in the opposite direction, will be equal to

$$Y\left(\frac{\partial a}{\partial x}-\tfrac{1}{2}\frac{\partial^2 a}{\partial x^2}dx\right)dS.$$

The resultant force in the X direction is consequently

$$Y\frac{\partial^2 a}{\partial x^2}dx\ dS.$$

This has to be equated to the product of the mass of the element and its acceleration, namely

$$\varrho\frac{\partial^2 a}{\partial t^2}dx\ dS.$$

On equating the two expressions and dividing both sides by the volume, $dx\,dS$, we get

$$Y\frac{\partial^2 a}{\partial x^2} = \varrho\frac{\partial^2 a}{\partial t^2}.$$

Consequently the velocity of propagation of such a disturbance along the rod is

$$v = \sqrt{\frac{Y}{\varrho}} \quad\quad . \quad . \quad . \quad . \quad . \quad (10\text{·}22)$$

The apparent discrepancy between this result and that expressed by formula (10·21), which undoubtedly represents correctly the velocity of propagation of purely longitudinal motions in a medium, is due to the fact that the propagation along the rod consists of longitudinal displacements associated with lateral contractions which travel along with them (see equation 10·04). This explanation can be verified by considering under what circumstances the longitudinal motions in the rod would be unaccompanied by lateral motions. This would be the case if Poisson's ratio (10·04) were zero, i.e. if

$$3k = 2n.$$

When this relation subsists between k and n, the velocities (10·21) and (10·22) are in fact identical as we should expect.

§ 10·3. Radial Strain in a Sphere

If the parts of the elastic medium are in equilibrium, and the body forces are negligible or zero, equation (10·2) becomes

$$n\nabla^2 a + \left(k + \frac{n}{3}\right)\frac{\partial}{\partial x}\,\mathbf{div}\,(a, \beta, \gamma) = 0$$

and with it are associated two similar equations

$$n\nabla^2 \beta + \left(k + \frac{n}{3}\right)\frac{\partial}{\partial y}\,\mathbf{div}\,(a, \beta, \gamma) = 0$$

$$n\nabla^2 \gamma + \left(k + \frac{n}{3}\right)\frac{\partial}{\partial z}\,\mathbf{div}\,(a, \beta, \gamma) = 0 \quad . \quad (10\text{·}3)$$

If now the strain consists in displacements w along radial lines from the origin of co-ordinates we have

$$a = \frac{x}{\mathbf{r}}w,$$

$$\beta = \frac{y}{\mathbf{r}}w,$$

$$\gamma = \frac{z}{\mathbf{r}}w, \quad\quad . \quad . \quad . \quad . \quad (10\text{·}301)$$

where $\mathbf{r} \equiv (x, y, z)$ and x, y and z are the co-ordinates (in its un-displaced condition) of the particle which suffers the displacement w. Then, remembering that $\dfrac{\partial \mathbf{r}}{\partial x} = \dfrac{x}{\mathbf{r}}$ and therefore $\dfrac{\partial \frac{1}{\mathbf{r}}}{\partial x} = -\dfrac{x}{\mathbf{r}^3}$ and $\dfrac{\partial w}{\partial x} = \dfrac{x}{\mathbf{r}}\dfrac{\partial w}{\partial \mathbf{r}}$, we obtain

$$\frac{\partial \alpha}{\partial x} = \left(\frac{1}{\mathbf{r}} - \frac{x^2}{\mathbf{r}^3}\right)w + \frac{x^2}{\mathbf{r}^2}\frac{\partial w}{\partial \mathbf{r}}$$

$$\frac{\partial \beta}{\partial y} = \left(\frac{1}{\mathbf{r}} - \frac{y^2}{\mathbf{r}^3}\right)w + \frac{y^2}{\mathbf{r}^2}\frac{\partial w}{\partial \mathbf{r}}$$

$$\frac{\partial \gamma}{\partial z} = \left(\frac{1}{\mathbf{r}} - \frac{z^2}{\mathbf{r}^2}\right)w + \frac{z^2}{\mathbf{r}^2}\frac{\partial w}{\partial}. \quad . \quad . \quad (10\cdot302)$$

Consequently

$$\mathbf{div}\,(\alpha, \beta, \gamma) = \frac{2}{\mathbf{r}}w + \frac{\partial w}{\partial \mathbf{r}} \quad . \quad . \quad . \quad (10\cdot31)$$

In a similar way it is easy to show that

$$\nabla^2 \alpha = \frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial^2 \alpha}{\partial z^2} = -\frac{2x}{\mathbf{r}^3}w + \frac{2x}{\mathbf{r}^2}\frac{\partial w}{\partial \mathbf{r}} + \frac{x}{\mathbf{r}}\frac{\partial^2 w}{\partial \mathbf{r}^2}. \quad (10\cdot32)$$

On substituting in $(10\cdot3)$ we get

$$n\left\{-\frac{2x}{\mathbf{r}^3}w + \frac{2x}{\mathbf{r}^2}\frac{dw}{dr} + \frac{x}{r}\frac{d^2w}{dr^2}\right\}$$

$$+ \left(k + \frac{n}{3}\right)\frac{\partial}{\partial x}\left\{\frac{2}{r}w + \frac{\partial w}{dr}\right\} = 0. \quad (10\cdot33)$$

If we now turn the axes of co-ordinates about the origin to make the X axis coincide with $\mathbf{r}$ or w, we shall have $x = \mathbf{r}$ and the differentiation $\dfrac{\partial}{\partial x}$ becomes $\dfrac{d}{dr}$, since for any function, ϕ of $\mathbf{r}$ only

$$\frac{\partial \phi}{\partial x} = \frac{d\phi}{dr} \cdot \frac{\partial r}{\partial x}$$

$$= \frac{x}{\mathbf{r}}\frac{d\phi}{dr} = \frac{d\phi}{dr}, \text{ when } r = x.$$

The equation $(10\cdot33)$ therefore simplifies to

$$\left(k + \frac{4n}{3}\right)\left\{\frac{d^2w}{dr^2} + \frac{2}{\mathbf{r}}\frac{dw}{dr} - \frac{2w}{\mathbf{r}^2}\right\} = 0,$$

or

$$\mathbf{r}^2\frac{d^2w}{d\mathbf{r}^2} + 2\mathbf{r}\frac{dw}{d\mathbf{r}} - 2w = 0 \quad . \quad . \quad (10\cdot34)$$

If we substitute $\mathbf{r}^n$ for w in this equation we find that it is satisfied provided n is a root of the equation

$$n(n-1) + 2n - 2 = 0$$

or
$$n^2 + n - 2 = 0 \qquad . \qquad (10·341)$$

Such an equation is called an **indicial equation.** Its roots in the present instance are $+1$ and -2. Therefore $\mathbf{r}$ and $\mathbf{r}^{-2}$ are particular solutions of the differential equation (**10·34**) and the general solution is

$$w = A\mathbf{r} + \frac{B}{\mathbf{r}^2} \qquad . \qquad . \qquad . \qquad . \qquad (10·35)$$

A and B being arbitrary constants. For the normal tension $t_{xx} = t_{rr}$ along a radial line through origin let us write $-p_r$, so that p_r is the corresponding pressure. We have now $\dfrac{\partial a}{\partial x} = \dfrac{\partial w}{\partial r}$ and $\dfrac{\partial \beta}{\partial y} = \dfrac{\partial \gamma}{\partial z} = \dfrac{w}{r}$. This latter relation follows at once from

$$\mathbf{div}\,(a, \beta, \gamma) = \frac{\partial a}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}$$

or
$$\frac{dw}{dr} + \frac{2w}{r} = \frac{dw}{dr} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}.$$

Therefore by (**10·02**)

$$-p_r = \left(k + \frac{4n}{3}\right)\frac{dw}{d\mathbf{r}} + 2\left(k - \frac{2n}{3}\right)\frac{w}{\mathbf{r}},$$

and on substituting for w the expression (**10·35**) we get

$$-p_r = 3kA - \frac{4nB}{r^3} \qquad . \qquad . \qquad . \qquad . \qquad (10·36)$$

If we write $-p_\tau$ for the tensions $t_{yy} = t_{zz}$ in directions perpendicular to $\mathbf{r}$ we shall have

$$-p_\tau = \left(k - \frac{2n}{3}\right)\frac{dw}{d\mathbf{r}} + \left(k + \frac{4n}{3}\right)\frac{w}{\mathbf{r}} + \left(k - \frac{2n}{3}\right)\frac{w}{\mathbf{r}}$$

or
$$-p_\tau = 3kA + \frac{2nB}{\mathbf{r}^3} \qquad . \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (10·361)$$

If we consider a spherical portion of the medium with its centre at the origin, it is evident that the constant B must be zero, otherwise the displacement, w, as well as the pressures p_r and p_τ would be infinite at the centre. In this case then $p_r' = p_\tau$ $= -3kA$, and we have a uniform pressure the corresponding dilatation being $3A$. Indeed the dilatation will in any case be constant and equal to $3A$, as will at once appear on substituting the expression (**10·35**) for w in equation (**10·31**).

We next consider a spherical shell, i.e. a portion of the medium enclosed between concentric spheres of radii r_1 (inner) and r_2, the common centre being at the origin. If p_{2r} is the pressure on the outside and p_{1r} on the interior, (10·36) gives the two equations

$$- p_{2r} = 3kA - \frac{4nB}{r_2{}^3},$$

$$- p_{1r} = 3kA - \frac{4nB}{r_1{}^3},$$

which enable us to determine A and B in terms of these two pressures and the elastic moduli. We find

$$A = \frac{r_1{}^3 p_{1r} - r_2{}^3 p_{2r}}{3k(r_2{}^3 - r_1{}^3)},$$

$$B = \frac{(p_{1r} - p_{2r})r_1{}^3 r_2{}^3}{4n(r_2{}^3 - r_1{}^3)},$$

and on substituting these expressions for A and B in (10·35), (10·36) and (10·361) we can evaluate the displacement and the pressures radial and transverse at any point in the interior of the shell.

The type of problem just solved is of practical importance, for instance in the measurement of the compressibility (i.e. the reciprocal of the bulk modulus) of liquids.

§ 10·4. Energy in a Strained Medium

Imagine a cylindrical element of volume with its axis parallel to the X axis. Let its length be l and cross-sectional area dS and suppose the state of stress in the medium is simply a tension t_{xx}. Then the work done in producing a displacement a of one end of the cylinder relative to the other will be

$$\int_0^a t_{xx} dS da,$$

since t_{xx} means the force per unit area. If the length of the cylinder be l this may be written

$$dS . l . \int t_{xx} d\left(\frac{a}{l}\right).$$

The volume of the element is ldS and when its dimensions are very small

$$\frac{a}{l} = \frac{\partial a}{\partial x} = S_{xx}$$

and hence the energy of strain is

$$\int_0^{S_{xx}} t_{xx} dS_{xx}$$

per unit volume.

If we use the principal axes of the strain (or stress, since we are dealing with an isotropic medium) as co-ordinate axes we find for the strain energy per unit volume

$$\int \{T_{xx} dS_{xx} + T_{yy} dS_{yy} + T_{zz} dS_{zz}\}.$$

Substituting e, f and g for S_{xx}, S_{yy} and S_{zz} respectively for the sake of abbreviation and replacing T_{xx}, T_{yy} and T_{zz} by the equivalent expressions in (**10·02**), we obtain

$$\int_0^{e, f, g} \{(Le + Mf + Mg)de + (Me + Lf + Mg)df$$
$$+ (Me + Mf + Lg)dg\},$$

in which

$$L = k + \frac{4n}{3},$$

$$M = k - \frac{2n}{3}.$$

This becomes
$$\tfrac{1}{2}L(e^2 + f^2 + g^2) + M(ef + fg + ge),$$
or

$$\tfrac{1}{2}k(e + f + g)^2 + \frac{n}{3}\{(e - f)^2 + (f - g)^2 + (g - e)^2\} \quad . \textbf{(10·4)}$$

This expression represents the strain energy per unit volume in terms of the principal elongations e, f and g and the moduli k and n.

§ 10·5. EQUATION OF CONTINUITY. PREVISION OF RELATIVITY

It will be remembered that a distinction was made between the co-ordinates (x, y, z), which refer to the positions of portions or elements of the medium in its undisplaced or undeformed condition, and the co-ordinates (ξ, η, ζ) which refer to actual or instantaneous positions at some instant t. In the present paragraph we are concerned with the latter co-ordinates only, but we shall represent them by (x, y, z) instead of (ξ, η, ζ). Having made this clear, let us proceed to find an expression for the mass of the medium which passes per second through a closed

surface from the interior outwards. Let $\mathbf{dS}$ (Fig. 10·5) represent an element of area of the closed surface, its vectorial arrow being (as usual) directed outwards. Let the direction of motion of the medium in the neighbourhood of dS at some instant t make an angle θ with that of the vector $\mathbf{dS}$ and suppose its velocity to be $\mathbf{c} \equiv (u, v, w)$. Obviously u, v and w are functions of x, y, z and t. Construct a cylinder with its axis parallel to $\mathbf{c}$ and with a cross-sectional area $dA = \mathbf{dS} \cos \theta$. It is not difficult to see that the mass of the medium passing through dS per second will be equal to that contained in a portion of the cylinder of length $\mathbf{c}$. If ϱ be the density of the medium this will be equal to

$$\varrho \mathbf{c} \, dA,$$
$$= \varrho \mathbf{c} \, \mathbf{dS} \cos \theta,$$
$$= (\varrho \mathbf{c}, \, \mathbf{dS}),$$

and therefore the total mass emerging through the whole surface per second will be

$$\iint (\varrho \mathbf{c}, \, \mathbf{dS}).$$

By the theorem of Gauss (§ 3) this is equal to

$$\iiint \left\{ \frac{\partial(\varrho u)}{\partial x} + \frac{\partial(\varrho v)}{\partial y} + \frac{\partial(\varrho w)}{\partial w} \right\} dx \, dy \, dz \quad . \quad \textbf{(10·5)}$$

But the mass leaving any element of volume $dx \, dy \, dz$ per second must be equal to

$$- \frac{\partial \varrho}{\partial t} \, dx \, dy \, dz,$$

and therefore (10·5) is equivalent to

$$- \iiint \frac{\partial \varrho}{\partial t} \, dx \, dy \, dz \quad . \quad . \quad . \quad \textbf{(10·501)}$$

On equating (10·5) and (10·501) we obtain

$$\iiint \left\{ \frac{\partial(\varrho u)}{\partial x} + \frac{\partial(\varrho v)}{\partial y} + \frac{\partial(\varrho w)}{\partial z} + \frac{\partial \varrho}{\partial t} \right\} dx \, dy \, dz = 0 \quad . \quad \textbf{(10·51)}$$

This result is true for any volume and therefore true when the volume is simply the element $dx \, dy \, dz$. We may therefore

FIG. 10·5

drop the symbols of integration $\iiint$ and so obtain the important result

$$\frac{\partial(\varrho u)}{\partial x} + \frac{\partial(\varrho v)}{\partial y} + \frac{\partial(\varrho w)}{\partial z} + \frac{\partial \varrho}{\partial t} = 0 \quad . \quad . \quad (10\cdot52)$$

This is called the **equation of continuity.**

We shall now turn back to equation (**10·2**), and give our attention to the case where the body force is zero. The equation therefore becomes

$$\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} = \varrho \frac{\partial^2 a}{\partial t^2}.$$

The x, y and z on the left are, as we have seen (§ 9·9), the instantaneous co-ordinates of the part of the medium considered and the t_{xx}, t_{xy}, t_{xz} are functions of these co-ordinates. On the other hand the a, in $\varrho \frac{\partial^2 a}{\partial t^2}$, on the right is regarded as a function of t and the co-ordinates of the medium in its *undisplaced* condition. We shall now express the acceleration in another way. It is of course equal to the increase in velocity $u_2 - u_1$ divided by the corresponding time $t_2 - t_1$, or, strictly speaking, the limit to which this ratio approaches as $t_2 - t_1$ is indefinitely decreased. Now if u is a function of t and the instantaneous co-ordinates x, y, z, $(u_2 - u_1)/(t_2 - t_1)$ becomes in the limit

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt},$$

or

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial w}{\partial z},$$

and consequently (**10·2**) may be written in the form

$$-\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z}\right) = \varrho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \quad (10\cdot521)$$

Now add to this equation

$$0 = u\left(\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho u)}{\partial x} + \frac{\partial(\varrho v)}{\partial y} + \frac{\partial(\varrho w)}{\partial z}\right),$$

which is simply the equation of continuity multiplied by u. We obtain

$$-\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z}\right) = \frac{\partial(\varrho u)}{\partial t} + \frac{\partial(\varrho u^2)}{\partial x} + \frac{\partial(\varrho uv)}{\partial y} + \frac{\partial(\varrho uw)}{\partial z},$$

or

$$\frac{\partial(p_{xx} + \varrho u^2)}{\partial x} + \frac{\partial(p_{xy} + \varrho uv)}{\partial y} + \frac{\partial(p_{xz} + \varrho uw)}{\partial z} + \frac{\partial(\varrho u)}{\partial t} = 0 \quad (10\cdot53)$$

We have so far spoken of the velocity (u, v, w) in terms which imply that every part of the medium within a sufficiently small volume element will have the velocity (u, v, w) or a velocity differing from it infinitesimally. The medium is however granular in constitution, and the individual particles or molecules will have velocities which differ widely from one another. What then is the meaning of the velocity (u, v, w) ? It is clear that when we associate this velocity with an element of volume $dx\,dy\,dz$ it can only mean the velocity of its centre of mass. Let m_s be the mass of a single molecule and (u_s, v_s, w_s) its velocity, and let (u_s', v_s', w_s') be its velocity *relative to the centre of mass of an element of volume within which it is situated*, then

$$u_s = u_s' + u,$$
$$v_s = v_s' + v,$$
$$w_s = w_s' + w.$$

Consider now the quantity

$$\Sigma m_s u_s v_s,$$

where the summation is extended over the unit volume, i.e. it is carried out over all the particles in an element of volume and the result divided by the volume. The sum

$$\Sigma m_s u_s v_s = \Sigma m_s (u_s' + u)\,(v_s' + v)$$
$$= \Sigma m_s u_s' v_s' + v \Sigma m_s u_s'$$
$$+ u \Sigma m_s v_s' + uv \Sigma m_s.$$

This reduces to

$$\Sigma m_s u_s v_s = \Sigma m_s u_s' v_s' + \varrho uv$$

because
$$\Sigma m_s u_s' = \Sigma m_s (u_s - u) = 0,$$
and
$$\Sigma m_s v_s' = \Sigma m_s (v_s - v) = 0,$$

by the definition of centre of mass.

Considerations exactly similar to those explained above in arriving at the mass $\varrho c\,dA$, passing per second through the area dA at right angles to the velocity **c** (Fig. 10·5) lead us to the conclusion that

$$\Sigma m_s u_s' v_s'\,dS_y$$

is the X component of the momentum which crosses the boundary dS_y (see Fig. 9·73) per second from the side a to the side b. It is therefore equal to the X component of the force exerted on the medium on the side b of dS_y by the medium on the side a. Consequently

$$\Sigma m_s u_s' v_s' = p_{xy},$$

and therefore
$$\Sigma m_s u_s v_s = p_{xy} + \varrho uv.$$

It is now evident that we may express (**10·53**) in the form

$$\frac{\partial(\Sigma m_s u_s{}^2)}{\partial x} + \frac{\partial(\Sigma m_s u_s v_s)}{\partial y} + \frac{\partial(\Sigma m_s u_s w_s)}{\partial z} + \frac{\partial(\Sigma m_s u_s)}{\partial t} = 0 \quad (\mathbf{10\cdot54})$$

To this we may add two equations similarly associated with the Y and Z axes.

The form of equation (**10·54**) suggests a four-dimensional divergence. This suggestion becomes still stronger if we multiply the last term above and below by a constant c with the dimensions of a velocity—we need not at present inquire whether any physical significance can be attached to c—and use the letter l for the distance ct. We thus obtain

$$\frac{\partial}{\partial x}(\Sigma m_s u_s{}^2) + \frac{\partial}{\partial y}(\Sigma m_s u_s v_s) + \frac{\partial}{\partial z}(\Sigma m_s u_s w_s) + \frac{\partial}{\partial l}(\Sigma m_s u_s c) = 0.$$

To this equation we may of course add

$$\frac{\partial}{\partial x}(\Sigma m_s v_s u_s) + \frac{\partial}{\partial y}(\Sigma m_s v_s{}^2) + \frac{\partial}{\partial z}(\Sigma m_s v_s w_s) + \frac{\partial}{\partial l}(\Sigma m_s v_s c) = 0,$$

$$\frac{\partial}{\partial x}(\Sigma m_s w_s u_s) + \frac{\partial}{\partial y}(\Sigma m_s w_s v_s) + \frac{\partial}{\partial z}(\Sigma m_s w_s{}^2) + \frac{\partial}{\partial l}(\Sigma m_s w_s c) = 0,$$

and the following fourth equation is suggested :

$$\frac{\partial}{\partial x}(\Sigma m_s c u_s) + \frac{\partial}{\partial y}(\Sigma m_s c v_s) + \frac{\partial}{\partial z}(\Sigma m_s c w_s)$$
$$+ \frac{\partial}{\partial l}(\Sigma m_s c^2) = 0 \qquad . \quad (\mathbf{10·55})$$

Now this fourth equation (**10·55**) is one we have already derived. It is in fact the equation of continuity since

$$\varrho u = \Sigma m_s u_s,$$
$$\varrho v = \Sigma m_s v_s,$$
$$\varrho w = \Sigma m_s w_s.$$

Equations (**10·55**) give us a prevision of the restricted or, as Einstein prefers to call it, the special theory of relativity, which draws space and time together into one continuum of a Euclidean character.

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LOVE : Mathematical Theory of Elasticity. See also the works of Thomson and Tait, Webster and Gray mentioned above (pp. 91, 120).

CHAPTER VIII

HYDRODYNAMICS

§ 10·6. EQUATIONS OF EULER AND LAGRANGE

WE now turn our attention to media for which $n = 0$. In such media the tangential stresses, such as $t_{xy} = -p_{xy}$, are zero and $p_{xx} = p_{yy} = p_{zz} = p$ (§ 10). Consequently the equations (**10·2**) take the form

$$-\frac{\partial p}{\partial x} + R_x = \varrho\frac{\partial^2\alpha}{\partial t^2},$$

$$-\frac{\partial p}{\partial y} + R_y = \varrho\frac{\partial^2\beta}{\partial t^2},$$

$$-\frac{\partial p}{\partial z} + R_z = \varrho\frac{\partial^2\gamma}{\partial t^2}, \quad . \quad . \quad . \quad . \quad (\mathbf{10\cdot6})$$

If the body force **R** is derivable from a potential, as is the case for example when it is due to gravity,

$$\mathbf{R} = -\varrho\,\mathbf{grad}\,V$$

$$R_x = -\varrho\frac{\partial V}{\partial x},$$

$$R_y = -\varrho\frac{\partial V}{\partial y},$$

$$R_z = -\varrho\frac{\partial V}{\partial z},$$

in which ϱ is the density of the medium ; and if in (**10·6**) we replace $\dfrac{\partial^2\alpha}{\partial t^2}$ by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

as in equation (**10·521**) we get

$$-\frac{1}{\varrho}\frac{\partial p}{\partial x} - \frac{\partial V}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

and two similar equations associated with the Y and Z axes

respectively.   The density, ϱ, is a function of the pressure, p, and therefore we may put $\dfrac{dp}{\varrho}$ in the form $d\varPi$ and so we get the equations

$$-\frac{\partial \varPi}{\partial x} - \frac{\partial V}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$

$$-\frac{\partial \varPi}{\partial y} - \frac{\partial V}{\partial y} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z},$$

$$-\frac{\partial \varPi}{\partial z} - \frac{\partial V}{z} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \qquad . \quad (10\cdot7)$$

These are **Euler's hydrodynamical equations.**   It is import-ant to have clear notions about the meanings of the variables, more especially the independent variables, which appear in these and other hydrodynamical equations.   We must regard x, y, and z as the co-ordinates of the centre, or centre of mass, of a small volume element at the instant t.   For the sake of brevity we shall say *particle* instead of *centre of mass of a small volume element*.   With this explanation we may describe x, y, and z as the co-ordinates of a particle of the medium at the instant t, or as the *instantaneous* co-ordinates of the particle.   It is how-ever sometimes convenient to use as independent variables the co-ordinates $(x_0,\ y_0,\ z_0)$ and the time, t, where $(x_0\ y_0\ z_0)$ give the position of the particle at some earlier instant, t_0 ; and we shall have to be on our guard against the error due to attaching the same meaning to $\partial/\partial t$ in the two cases.   If ψ is some function of the four independent variables, we shall adopt for partial differentiation with respect to t the following notation :

$$\frac{\partial \psi}{\partial t} \quad \text{when}\ \ \psi = \text{function}\ \ (x,\ y,\ z,\ t),$$

and $\qquad\qquad \dfrac{D\psi}{Dt} \quad$ when $\ \ \psi = \text{function}\ \ (x_0,\ y_0,\ z_0,\ t) \quad . \quad (10\cdot701)$

The differential quotient $\dfrac{D\psi}{Dt}$ therefore means a partial differen-tiation of ψ with respect to t when x_0, y_0 and z_0 are not varied. It therefore means the limiting value of

$$\frac{\psi_2 - \psi_1}{t_2 - t_1}$$

where ψ_2 and ψ_1 are the values of ψ for *the same particle* $(x_0,\ y_0$ and z_0 being unvaried) at the times t_2 and t_1 respectively.   But we have already seen (§ 10·5) that this limiting value is

$$\frac{\partial \psi}{\partial t} + u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y} + w\frac{\partial \psi}{\partial w}.$$

Therefore

$$\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y} + w\frac{\partial \psi}{\partial z}. \quad . \quad (10 \cdot 702)$$

Suppose for example that $\psi = x$. Our formula becomes

$$\frac{Dx}{Dt} = \frac{\partial x}{\partial t} + u\frac{\partial x}{\partial x} + v\frac{\partial x}{\partial y} + w\frac{\partial x}{\partial z}.$$

But $x\ y\ z$ and t are *independent* variables ; therefore

$$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial y} = \frac{\partial x}{\partial z} = 0$$

and

$$\frac{\partial x}{\partial x} = 1,$$

consequently

$$\frac{Dx}{Dt} = u,$$

as is otherwise evident from the definition of $\dfrac{D}{Dt}$ given above.

Obviously Euler's equations may be expressed in the form

$$-\frac{\partial}{\partial x}(\Pi + V) = \frac{Du}{Dt},$$

$$-\frac{\partial}{\partial y}(\Pi + V) = \frac{Dv}{Dt},$$

$$-\frac{\partial}{\partial z}(\Pi + V) = \frac{Dw}{Dt}. \quad . \quad . \quad . \quad (10 \cdot 703)$$

If we multiply these equations by $\dfrac{\partial x}{\partial x_0}$, $\dfrac{\partial y}{\partial x_0}$ and $\dfrac{\partial z}{\partial x_0}$ respectively —x, y and z can be regarded as functions of x_0, y_0, z_0 and t— and add, we obtain

$$-\left\{\frac{\partial}{\partial x}(\Pi + V)\frac{\partial x}{\partial x_0} + \frac{\partial}{\partial y}(\Pi + V)\frac{\partial y}{\partial x_0} + \frac{\partial}{\partial z}(\Pi + V)\frac{\partial z}{\partial x_0}\right\}$$

$$= \frac{Du}{Dt}\frac{\partial x}{\partial x_0} + \frac{Dv}{Dt}\frac{\partial y}{\partial x_0} + \frac{Dw}{Dt}\frac{\partial z}{\partial x_0},$$

or

$$-\frac{\partial}{\partial x_0}(\Pi + V) = \frac{Du}{Dt}\frac{\partial x}{\partial x_0} + \frac{Dv}{Dt}\frac{\partial y}{\partial x_0} + \frac{Dw}{Dt}\frac{\partial z}{\partial x_0},$$

to which we may of course add

$$-\frac{\partial}{\partial y_0}(\Pi + V) = \frac{Du}{Dt}\frac{\partial x}{\partial y_0} + \frac{Dv}{Dt}\frac{\partial y}{\partial y_0} + \frac{Dw}{Dt}\frac{\partial z}{\partial y_0},$$

$$-\frac{\partial}{\partial z_0}(\Pi + V) = \frac{Du}{Dt}\frac{\partial x}{\partial z_0} + \frac{Dv}{Dt}\frac{\partial y}{\partial z_0} + \frac{Dw}{Dt}\frac{\partial z}{\partial z_0}. \quad (10 \cdot 71)$$

These are the **hydrodynamical equations of Lagrange.**

The complete expression for $\dfrac{\partial \psi}{\partial x_0}$ being

$$\frac{\partial \psi}{\partial x_0} = \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial x_0} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial x_0} + \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial x_0} + \frac{\partial \psi}{\partial t}\frac{\partial t}{\partial x_0},$$

it seems as if we ought to have written the left-hand side of (for example) the first equation (10·71) as

$$-\frac{\partial(\varPi + V)}{\partial x_0} + \frac{\partial(\varPi + V)}{\partial t}\frac{\partial t}{\partial x_0}.$$

But we have to remember that t and x_0 are independent variables and therefore

$$\frac{\partial t}{\partial x_0} = 0.$$

Such an expression as $\dfrac{Du}{Dt}\dfrac{\partial x}{\partial x_0}$ may be written in the form

$$\frac{D}{Dt}\left(u\frac{\partial x}{\partial x_0}\right) - u\frac{D}{Dt}\left(\frac{\partial x}{\partial x_0}\right),$$

or, since t and x_0 are independent, we may interchange $\dfrac{D}{Dt}$ and $\dfrac{\partial}{\partial x_0}$, thus obtaining

$$\frac{Du}{Dt}\cdot\frac{\partial x}{\partial x_0} = \frac{D}{Dt}\left(u\frac{\partial x}{\partial x_0}\right) - u\frac{\partial}{\partial x_0}\left(\frac{Dx}{Dt}\right),$$

or
$$\frac{Du}{Dt}\frac{\partial x}{\partial x_0} = \frac{D}{Dt}\left(u\frac{\partial x}{\partial x_0}\right) - \frac{\partial}{\partial x_0}(\tfrac{1}{2}u^2),$$

similarly
$$\frac{Dv}{Dt}\frac{\partial y}{\partial x_0} = \frac{D}{Dt}\left(v\frac{\partial y}{\partial x_0}\right) - \frac{\partial}{\partial x_0}(\tfrac{1}{2}v^2),$$

and
$$\frac{Dw}{Dt}\frac{\partial z}{\partial x_0} = \frac{D}{Dt}\left(w\frac{\partial z}{\partial x_0}\right) - \frac{\partial}{\partial x_0}(\tfrac{1}{2}w^2) \quad . \quad (10·711)$$

Applying this result in equation (10·71) we get

$$-\frac{\partial}{\partial x_0}(\varPi + V) = \frac{D}{Dt}\left(u\frac{\partial x}{\partial x_0} + v\frac{\partial y}{\partial x_0} + w\frac{\partial z}{\partial x_0}\right) - \frac{\partial}{\partial x_0}(\tfrac{1}{2}c^2)$$

or

$$-\frac{\partial}{\partial x_0}(\varPi + V - \tfrac{1}{2}c^2) = \frac{D}{Dt}\left(u\frac{\partial x}{\partial x_0} + v\frac{\partial y}{\partial x_0} + w\frac{\partial z}{\partial x_0}\right) \quad . \quad (10·712)$$

If we multiply both sides of this last equation by Dt and integrate between the limits t_0 and t ($x_0\ y_0\ z_0$ being kept constant of course, so that the function $\varPi + V - \tfrac{1}{2}c^2$ refers all the time to the same material in the course of its motion) we obtain

$$-\frac{\partial}{\partial x_0}\int_{t_0}^{t}(\varPi + V - \tfrac{1}{2}c^2)Dt = u\frac{\partial x}{\partial x_0} + v\frac{\partial y}{\partial x_0} + w\frac{\partial w}{\partial x_0} - u_0,$$

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since
$$\frac{\partial x_0}{\partial x_0} = 1 \text{ and } \frac{\partial y_0}{\partial x_0} = \frac{\partial z_0}{\partial x_0} = 0.$$

If we represent the integral on the left by χ, this becomes
$$-\frac{\partial}{\partial x_0}\chi = u\frac{\partial x}{\partial x_0} + v\frac{\partial y}{\partial x_0} + w\frac{\partial z}{\partial x_0} - u_0,$$

and we may similarly derive
$$-\frac{\partial}{\partial y_0}\chi = u\frac{\partial x}{\partial y_0} + v\frac{\partial y}{\partial y_0} + w\frac{\partial z}{\partial y_0} - v_0,$$

$$-\frac{\partial}{\partial z_0}\chi = u\frac{\partial x}{\partial z_0} + v\frac{\partial y}{\partial z_0} + w\frac{\partial z}{\partial z_0} - w_0 \quad . \quad . \quad (10 \cdot 72)$$

These are **Weber's hydrodynamical equations.** Let $(x_0 + \delta x_0,$ $y_0 + \delta y_0, z_0 + \delta z_0)$ be the position of a particle in the neighbourhood of (x_0, y_0, z_0) at the same time t_0, and multiply Weber's equations by δx_0, δy_0 and δz_0 respectively and add. We thus find

$$-\left\{\left(\frac{\partial \chi}{\partial x_0} - u_0\right)\delta x_0 + \left(\frac{\partial \chi}{\partial y_0} - v_0\right)\delta y_0 + \left(\frac{\partial \chi}{\partial z_0} - w_0\right)\delta z_0\right\}$$
$$= u\delta x + v\delta y + w\delta z \quad . \quad . \quad (10 \cdot 721)$$

If at the time t_0 a velocity potential ϕ_0 exists, i.e. if everywhere
$$u_0 = -\frac{\partial \phi_0}{\partial x_0},$$

$$v_0 = -\frac{\partial \phi_0}{\partial y_0},$$

$$w_0 = -\frac{\partial \phi_0}{\partial z_0},$$

equation $(10 \cdot 721)$ becomes
$$-\left\{\frac{\partial}{\partial x_0}(\chi + \phi_0)\delta x_0 + \frac{\partial}{\partial y_0}(\chi + \phi_0)\delta y_0 + \frac{\partial}{\partial z_0}(\chi + \phi_0)\delta z_0\right\}$$
$$= u\delta x + v\delta y + w\delta z;$$

or since
$$\delta x_0 = \frac{\partial x_0}{\partial x}\delta x + \frac{\partial x_0}{\partial y}\delta y + \frac{\partial x_0}{\partial z}\delta z,$$

$$\delta y_0 = \frac{\partial y_0}{\partial x}\delta x + \frac{\partial y_0}{\partial y}\delta y + \frac{\partial y_0}{\partial z}\delta z,$$

$$\delta z_0 = \frac{\partial z_0}{\partial x}\delta x + \frac{\partial z_0}{\partial y}\delta y + \frac{\partial z_0}{\partial z}\delta z,$$

$$-\left\{\frac{\partial}{\partial x}(\chi + \phi_0)\delta x + \frac{\partial}{\partial y}(\chi + \phi_0)\delta y + \frac{\partial}{\partial z}(\chi + \phi_0)dz\right\}$$
$$= u\delta x + v\delta y + w\delta z.$$

Hence in such a case

$$u = -\frac{\partial}{\partial x}(\chi + \phi_0), \quad = -\frac{\partial \phi}{\partial x}$$

$$v = -\frac{\partial}{\partial y}(\chi + \phi_0), \quad = -\frac{\partial \phi}{\partial y}$$

$$w = -\frac{\partial}{\partial z}(\chi + \phi_0) \quad = -\frac{\partial \phi}{\partial z} \quad . \quad . \quad (10\text{·}73)$$

This means that if at any instant t_0 a velocity potential ϕ_0 exists there will always be a velocity potential $\phi = \chi + \phi_0$.

§ 10·8. ROTATIONAL AND IRROTATIONAL MOTION

In equations (9·622) a small rotation of the medium is represented by (q_x, q_y, q_z) and the corresponding small displacements by (α, β, γ). If both sides of the equations are divided by the short interval of time during which the rotation and displacements are effected they become

$$2\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},$$

$$2\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},$$

$$2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad . \quad . \quad . \quad (10\text{·}8)$$

where $(\omega_x, \omega_y, \omega_z)$ is the angular velocity of the medium. We may also arrive at this result in the following way. By the theorem of Stokes

$$\oint (u\delta x + v\delta y + w\delta z) = \int\int (\mathbf{curl\ c, dS}),$$

the integration on the left extending round any closed loop in the medium at the same instant, t. Now suppose a motion of pure rotation with an angular velocity $\boldsymbol{\omega}$ to exist in the neighbourhood of some point (x, y, z) and imagine the closed loop to be a small circle of radius r having its axis coincident with the axis of rotation. The line integral on the left becomes

$$2\pi r\mathbf{c},$$

while the surface integral on the right becomes

$$\mathbf{curl\ c}.\pi r^2.$$

Therefore

$$2\pi r\ \mathbf{c} = \mathbf{curl\ c}.\pi r^2,$$

$$2\frac{\mathbf{c}}{r} = \mathbf{curl\ c}.$$

But c/r is the angular velocity.   Therefore

$$2\boldsymbol{\omega} = \mathbf{curl\ c}.$$

This is equivalent to (**10·8**).

When $\boldsymbol{\omega} = 0$ we speak of the motion as **irrotational**.   It should be observed that a mass of fluid may be revolving about some axis while its motion is nevertheless *irrotational* in the sense defined above.   This is illustrated in Fig. 10·8.   The case (*a*) represents irrotational motion, the true **rotational** motion
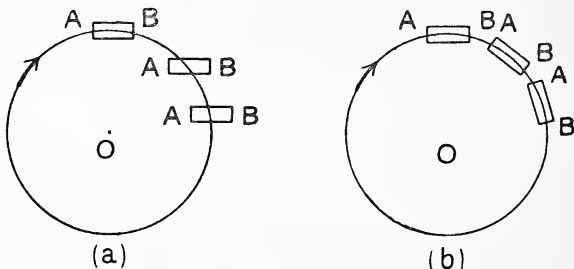

FIG. 10·8

being shown in (*b*).   In the former case **curl c** is zero notwithstanding the fact that the fluid may be said to revolve about an axis O.   Clearly the motion will be irrotational if a velocity potential exists, since the curl of a gradient is zero (**2·431**).

§ 10·9.   THEOREM OF BERNOULLI

Turning to the first of Euler's equations (**10·7**), let us subtract from it the identical equation

$$\frac{\partial}{\partial x}(\tfrac{1}{2}\mathbf{c}^2) = \tfrac{1}{2}\frac{\partial}{\partial x}(u^2 + v^2 + w^2)$$

or

$$\frac{\partial}{\partial x}(\tfrac{1}{2}\mathbf{c}^2) = u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}.$$

We thus get

$$-\frac{\partial}{\partial x}(\varPi + V + \tfrac{1}{2}\mathbf{c}^2) = \frac{\partial u}{\partial t} - 2v\omega_z + 2w\omega_y,$$

and, in a similar way, we find

$$-\frac{\partial}{\partial y}(\varPi + V + \tfrac{1}{2}\mathbf{c}^2) = \frac{\partial v}{\partial t} - 2w\omega_x + 2u\omega_z,$$

$$-\frac{\partial}{\partial z}(\varPi + V + \tfrac{1}{2}\mathbf{c}^2) = \frac{\partial w}{\partial t} - 2u\omega_y + 2v\omega_x. \qquad \textbf{(10·9)}$$

If the motion is **stationary**, i.e. if the state of affairs at any point (x, y, z) remains unchanged during the motion, all the differential quotients $\partial/\partial t$ are zero, hence

$$-\frac{\partial}{\partial x}(\Pi + V + \tfrac{1}{2}\mathbf{c}^2) = 2w\omega_y - 2v\omega_z,$$

$$-\frac{\partial}{\partial y}(\Pi + V + \tfrac{1}{2}\mathbf{c}^2) = 2u\omega_z - 2w\omega_x,$$

$$-\frac{\partial}{\partial z}(\Pi + V + \tfrac{1}{2}\mathbf{c}^2) = 2v\omega_x - 2u\omega_y \quad . \quad (10\text{·}901)$$

If the motion is irrotational as well, $\omega_x = \omega_y = \omega_z = 0$ and consequently

$$\Pi + V + \tfrac{1}{2}\mathbf{c}^2 = \text{constant} \quad . \quad . \quad . \quad (10\text{·}91)$$

This result, known as **Bernoulli's theorem,** was given by Daniel Bernoulli in his *Hydrodynamica* (1738). Even if the motion is rotational Bernoulli's theorem will still be true for all points on the same **stream line**, that is to say $\Pi + V + \tfrac{1}{2}\mathbf{c}^2$ will remain constant along the path of a particle of the fluid, provided of course the motion is stationary. This can be shown in the following way : The equations of a stream line are obviously $dx : dy : dz = u : v : w$, or

$$dx = Au,$$
$$dy = Av,$$
$$dz = Aw. \quad . \quad . \quad . \quad (10\text{·}902)$$

Multiply equations ($10\text{·}901$) respectively by dx, dy and dz and make use of ($10\text{·}902$). We find

$$-d(\Pi + V + \tfrac{1}{2}\mathbf{c}^2) = 2A\,\{uw\omega_y - uv\omega_z$$
$$+ vu\omega_z - vw\omega_x + wv\omega_x - wu\omega_y\}.$$

The right-hand side is identically zero and therefore

$$\Pi + V + \tfrac{1}{2}\mathbf{c}^2 = \text{constant}.$$

If the fluid be incompressible ($\varrho = \text{constant}$) and V the gravitational potential, Bernoulli's theorem assumes the familiar form

$$\frac{p}{\varrho} + gh + \tfrac{1}{2}\mathbf{c}^2 = \text{constant} \quad . \quad . \quad . \quad (10\text{·}91)$$

h being the height of a point in the fluid measured from a fixed plane of reference, p the pressure at that point and $\mathbf{c}$ the velocity of the fluid.

The formula of Torricelli for the velocity with which a liquid emerges, under gravity, through a small orifice in a wide jar is an immediate consequence of Bernoulli's theorem. Let the .

fixed plane of reference be that in which the orifice, A, is situated
(Fig. 10·9 (a)) and consider first a point in the surface B of the
liquid, the area of which we shall suppose is very great com-
pared with the cross-section of the orifice.   The velocity at B
is practically zero, therefore the constant quantity of (**10·91**)
reduces to

$$\frac{B}{\varrho} + gh,$$

where B is the atmospheric pressure.   At the orifice on the
other hand h is zero, the pressure is again atmospheric and the
liquid has some velocity $\mathbf{c}$, therefore we have for the same
constant

$$\frac{B}{\varrho} + \tfrac{1}{2}\mathbf{c}^2.$$

On equating the two expressions we find

$$\mathbf{c}^2 = 2gh.$$

This result can only be approximately true for any other than
an ideal liquid, since the hydrodynamical equations from which it
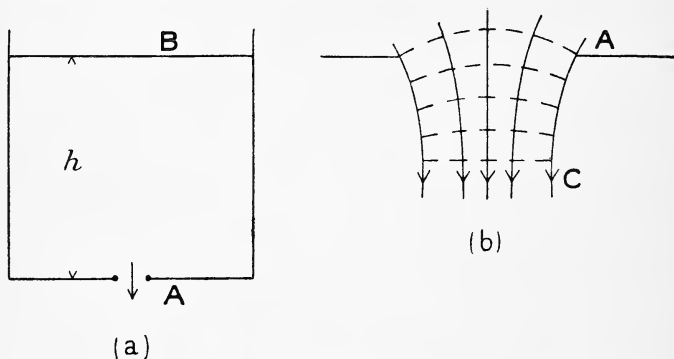
Fig. 10·9

has been derived entirely ignore viscosity.   It will be observed that
the deduction is only valid for points at the surface of the emerg-
ing jet, since it is only there we may assume the pressure to be
atmospheric.   If the orifice is very small the velocity will be
practically the same all round the jet, whether the orifice is in
the side of the vessel or at the bottom, as shown in Fig. 10·9.
The stream lines converge in the neighbourhood of the orifice
(see Fig. 10·9 (b)) until at a place C, a short distance outside
the vessel, the cross-sectional area of the jet is reduced, as will
be proved, to half that of the orifice.   This is the **vena contracta**.

Here obviously the stream lines become parallel and consequently the surfaces of constant potential, indicated in the figure by broken lines, are also parallel to one another and perpendicular to the axis of the jet. Thus the potential gradient, and therefore the velocity, will be constant in the vena contracta. If A be the cross-sectional area of the orifice and A' that of the vena contracta, the force exerted at the orifice on the emerging liquid must be equal to $A\varrho gh$. On the other hand it must be equal to the rate at which momentum passes through A and therefore through A', if we neglect momentum produced outside the orifice by gravity. But the volume of liquid passing through A' per second is $A'\mathbf{c}$ and the momentum per unit volume is $\varrho\mathbf{c}$. Hence the momentum passing through A' per second is $A'\varrho\mathbf{c}^2$. Consequently

$$A\varrho gh = A'\varrho\mathbf{c}^2$$

or

$$\mathbf{c}^2 = \frac{A}{A'}\,gh.$$

On comparing this with Torricelli's formula we see that $A = 2A'$.

§ 11. The Velocity Potential

When a velocity potential exists, i.e. when the velocity is a gradient, it is evident that the motion of the fluid is irrotational, since $2\boldsymbol{\omega}$ is equal to the curl of the velocity, and the curl of a gradient is zero (2·431). Conversely if the motion of the fluid is everywhere irrotational, i.e. if $\boldsymbol{\omega}$ is everywhere zero, the line integral

$$\oint(u\delta x + v\delta y + w\delta z),$$

taking round a closed loop in the fluid at some definite instant t will be zero by the theorem of Stokes. Hence the integral

$$\int_A^C (u\delta x + v\delta y + w\delta z) \text{ from a point } A \text{ to another point } C \text{ is}$$

independent of the path or

$$\int_{ABC} (u\delta x + v\delta y + w\delta z) = \int_{ADC} (u\delta x + v\delta y + w\delta z),$$

(see § 5·1 and Fig. 5·1). Whence it follows that $\mathbf{c} \equiv (u, v, w)$ is a gradient ; in other words a velocity potential exists. This type of field vector is called a **lamellar** vector and the corresponding fluid motion is called **lamellar motion**₀

If the fluid is incompressible (ϱ constant) the equation of continuity (**10·52**) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

or

$$\mathbf{div}\ \mathbf{c} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \textbf{(11)}$$

and consequently, by (**10·73**),

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial \phi}{\partial z^2} = 0 \quad . \quad . \quad . \quad . \quad \textbf{(11·01)}$$

This is **Laplace's equation.** We shall see later that it is also the equation for the potential in an electrostatic field, in regions where there is no charge, when ϕ is the potential; and therefore problems of lamellar flow in an incompressible fluid are mathematically identical with electrostatic problems in regions free from electric charges.

For the sake of argument let us imagine fluid to be created at one point (which we may conveniently take to be the origin of rectangular co-ordinates) at a constant rate of Q cubic centimetres per second. Since ϱ is constant Q cubic centimetres will then pass per second through any spherical surface of radius $\mathbf{r}$ with its centre at the origin. This of course will be true of any surface completely enclosing the origin, on the assumption we are making that the density ϱ is constant. Hence the volume passing through any square centimetre of the spherical surface per second will be $Q/4\pi \mathbf{r}^2$. This means

$$\mathbf{c} = \frac{Q}{4\pi \mathbf{r}^2}.$$

We have therefore an inverse square law for $\mathbf{c}$, which, being the negative gradient of the velocity potential, corresponds to the field intensity in electrostatics. The point source of the fluid corresponds exactly to a point charge of electricity equal to Q in suitable units. Obviously the velocity potential is $\dfrac{Q}{4\pi \mathbf{r}}$, so

that $\dfrac{1}{\mathbf{r}}$ is a solution of (**11·01**). This has already been proved in § 3·1. Applying the method of § 3·1 to find the values of n for which $\mathbf{r}^n$ is a solution of (**11·01**), we get

$$\nabla^2(\mathbf{r}^n) = \{3n + (n-2)n\}\mathbf{r}^{n-2}$$

Therefore $n^2 + n = 0,$

or $n = -1 \text{ or } 0,$

and the corresponding solutions are

$$\phi = \text{constant},$$

and

$$\phi = \frac{\text{constant}}{\mathbf{r}} \quad . \quad . \quad . \quad . \quad . \quad \textbf{(11·02)}$$

We can derive an unlimited number of solutions from (11·02) by differentiation. We have

$$\frac{\partial}{\partial x}\nabla^2\phi = 0$$

and, in consequence of the independence of the variables, x, y, z,

$$\nabla^2\left(\frac{\partial\phi}{\partial x}\right) = 0.$$

Therefore if ϕ_1 is a solution of $\nabla^2\phi = 0$, so is $\frac{\partial\phi_1}{\partial x}$. Hence among the particular solutions of $\nabla^2\phi = 0$ are included the following :

$$\frac{\partial^{a+b+c}\left(\frac{1}{\mathbf{r}}\right)}{\partial x^a \partial y^b \partial z^c} \qquad . \quad . \quad . \quad . \quad . \quad (11\cdot03)$$

where a, b and c are any positive integers.

Instead of imagining a point source, let us suppose the fluid to be created at constant rate over an extended region. This means that we are giving up within this region the equation

$$\mathbf{div}\ \mathbf{c} = 0,$$

and therefore also $\qquad \nabla^2\phi = 0.$

If ε is the volume of fluid created per second in one cubic centimetre. The volume Q created per second in a sphere of radius $\mathbf{r}$ will be

$$Q = \varepsilon\frac{4\pi}{3}\mathbf{r}^3.$$

If we assume radial symmetry the velocity at points on the surface of the sphere will be

$$\mathbf{c} = \frac{Q}{4\pi\mathbf{r}^2} = \frac{\varepsilon}{3}\mathbf{r},$$

or $\qquad\qquad \mathbf{c} \equiv \left(\frac{\varepsilon}{3}x,\ \frac{\varepsilon}{3}y,\ \frac{\varepsilon}{3}z\right) \equiv (u,\ v,\ w)$

and therefore

$$\mathbf{div}\ \mathbf{c} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

or $\qquad\qquad \mathbf{div}\ \mathbf{c} = \varepsilon,$

or otherwise expressed

$$\nabla^2\phi = -\varepsilon \qquad . \quad . \quad . \quad . \quad . \quad (11\cdot04)$$

This result, known as **Poisson's equation**, will still hold even if the assumption of radial symmetry is dropped since any *additional* velocity, $\mathbf{c}'$, which we may imagine to be superposed on that possessing radial symmetry, is bound to conform to

$$\mathbf{div}\ \mathbf{c}' = 0$$

§ 11·1.   KINETIC ENERGY IN A FLUID

The kinetic energy in a given volume of the fluid will be

$$\tfrac{1}{2}\int\int\int \varrho \mathbf{c}^2 \, dx \, dy \, dz,$$

the integration being extended over the whole volume in question. If the fluid is incompressible (ϱ constant), and if a velocity potential exists, this becomes

$$T = \tfrac{1}{2}\varrho \int\int\int (\mathbf{grad}\ \phi)^2 \, dx \, dy \, dz.$$

If we write $U = V = \phi$ in equation (3·1) we find

$$T = -\tfrac{1}{2}\varrho \int\int\int \phi \bigtriangledown^2 \phi \, dx \, dy \, dz + \frac{\varrho}{2}\int\int (\phi\ \mathbf{grad}\ \phi,\ \mathbf{dS}),$$

and since $\bigtriangledown^2 \phi = 0$

$$T = \frac{\varrho}{2}\int\int (\phi\ \mathbf{grad}\ \phi,\ \mathbf{dS}) \quad . \quad . \quad . \quad (11\text{·}1)$$

the integration extending over the bounding surface or surfaces. Or we may express it in the form

$$T = \frac{\varrho}{2}\int\int \phi \frac{\partial \phi}{\partial n} \mathbf{dS} \quad . \quad . \quad . \quad . \quad (11\text{·}15)$$

where (§ 3·1) $\dfrac{\partial}{\partial n}$ means differentiation in the direction of the outward normal to the surface.

§ 11·2.   MOTION OF A SPHERE THROUGH AN INCOMPRESSIBLE FLUID

We shall now study the steady irrotational motion in an incompressible fluid through which a sphere is moving with a constant velocity. Let the centre of the sphere travel along the Z axis with the velocity $\mathbf{c}_0$, so that $\mathbf{c}_0 \equiv (0, 0, c_0)$. We may obviously take the velocity of the fluid at points very far away from the sphere to be zero. If we now imagine a velocity $-\mathbf{c}_0$ superposed on the sphere and fluid, the former will remain at rest, and we may choose that its centre is at the origin of co-ordinates; while the distant parts of the fluid will have the velocity

$$u = 0,$$
$$v = 0,$$
$$w = -c_0. \quad . \quad . \quad . \quad . \quad (11\text{·}2)$$

We now inquire about the velocity potential. Apart from a constant it must have the value

$$\phi = c_0 z \quad . \quad . \quad . \quad . \quad . \quad (11\text{·}201)$$

at distant points, in order to give the velocity (**11·2**). Near the surface of the sphere ϕ must conform to the condition

$$\frac{\partial \phi}{\partial r} = 0 \quad . \quad . \quad . \quad . \quad . \quad (11 \cdot 202)$$

on substituting R, the radius of the sphere, for r, since the component of the fluid velocity in directions normal to the surface of the sphere must necessarily be zero at the surface. In addition to the conditions (**11·201**) and (**11·202**) ϕ must of course satisfy equation (**11·01**). The particular solution $\phi = \dfrac{1}{r}$ does not help us, because of its radial symmetry. We want a solution with the axial type of symmetry and this at once suggests

$$\phi = \frac{\partial}{\partial z}\left(\frac{A}{r}\right) \quad . \quad . \quad . \quad . \quad (11 \cdot 203)$$

where A is some constant. The sum of the particular solutions (**11·201**) and (**11·203**) will also be a solution of Laplace's equation. We therefore try

$$\phi = c_0 z + \frac{\partial}{\partial z}\left(\frac{A}{r}\right)$$

or

$$\phi = c_0 z - \frac{Az}{r^3} \quad . \quad . \quad . \quad . \quad . \quad (11 \cdot 21)$$

This satisfies Laplace's equation and the condition (**11·2**) for the motion of the distant parts of the fluid, and we have still to inquire if the remaining condition (**11·202**)

$$\left| \frac{\partial \phi}{\partial r} \right|_{r=R} = 0$$

can be satisfied by giving A a suitable value. We can put ϕ in the form

$$\phi = c_0 r \frac{z}{r} - \frac{A}{r^2}\frac{z}{r},$$

or

$$\phi = \left(c_0 r - \frac{A}{r^2}\right)\cos\theta,$$

where θ is the angle between the Z axis and the radial line from the centre of the sphere through the point (x, y, z). The differentiation $\dfrac{\partial}{\partial r}$ means a differentiation subject to the condition that $\cos\theta$ is not changed; therefore

$$\frac{\partial \phi}{\partial r} = \left(c_0 + \frac{2A}{r^3}\right)\cos\theta,$$

and at the surface of the sphere this becomes

$$\left.\left|\frac{\partial \phi}{\partial r}\right|\right._{r=R} = \left(c_0 + \frac{2A}{R^3}\right)\cos\theta.$$

This will vanish if

$$A = -\frac{c_0 R^3}{2},$$

so that the appropriate expression for ϕ is

$$\phi = c_0 z + \frac{c_0 R^3}{2r^3}z \quad . \quad . \quad . \quad . \quad (11 \cdot 22)$$

This is the velocity potential for the case where the sphere is at rest with its centre at the origin, and the distant parts of the fluid have the velocity $(11 \cdot 2)$. If we now superpose on the whole system the velocity

$$u = 0,$$
$$v = 0,$$
$$w = c_0,$$

we have our original problem again, and ϕ becomes

$$\phi = c_0 z + \frac{c_0 R^3 z}{2r^3} - c_0 z,$$

or

$$\phi = \frac{c_0 R^3 z}{2r^3} \quad . \quad . \quad . \quad . \quad . \quad (11 \cdot 23)$$

It is important to note that this expression represents the velocity potential only at the instant $(t = 0)$ when the centre of the sphere is at the origin of co-ordinates.

We are now able to work out an expression for the kinetic energy T_f, of the fluid. Equation $(11 \cdot 15)$ gives us

$$T_f = \tfrac{1}{2}\varrho \int\int \frac{c_0 R^3 z}{2r^3} \frac{\partial}{\partial n}\left\{\frac{c_0 R^3 z}{2r^3}\right\}dS \quad . \quad . \quad (11 \cdot 24)$$

The integration has to be extended over the boundary of the fluid. We may think of the fluid as bounded by the sphere of radius R with its centre at the origin, and by a sphere of infinite radius with its centre at the origin. As the differentiation $\partial/\partial n$ is in the direction of the normal *outwards from the fluid*, it will be equivalent to $\partial/\partial r$ on the large sphere and to $-\partial/\partial r$ on the surface of radius R. In either differentiation $\cos\theta = \dfrac{z}{r}$ is constant. It is convenient to use as variables r, θ and ϕ, instead of x, y and z, where ϕ is the angle between any plane containing the Z axis and some fixed plane containing the Z axis, for example the XZ plane. We may suppose ϕ measured in the direction of rotation of a

screw which is travelling along the Z axis in a positive sense. The co-ordinates r, θ and ϕ are known as polar co-ordinates and are related to x, y and z in the following way :

$$z = r \cos \theta,$$
$$x = r \sin \theta \cos \phi,$$
$$y = r \sin \theta \sin \phi, \quad . \quad . \quad . \quad . \quad (11\cdot241)$$

If on a sphere of radius r we vary θ and keep ϕ constant we shall have a great circle. Collectively these circles will be like circles of longitude, and will have a common diameter contained in the Z axis. On the other hand if we vary ϕ and keep θ constant we shall get a circle (like a circle of latitude) with a radius $r \sin \theta$. A small element of area, dS, bounded by θ, $\theta + d\theta$, ϕ and $\phi + d\phi$ will be equal to

$$dS = r d\theta . r \sin \theta \, d\phi$$
or $$dS = r^2 \sin \theta \, d\theta \, d\phi \quad . \quad . \quad . \quad (11\cdot242)$$

Introducing the new co-ordinates in $(11\cdot24)$ we get

$$T_f = \tfrac{1}{2}\varrho \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \frac{c_0 R^3}{2r^2} \cos \theta \frac{\partial}{\partial n}\left\{\frac{c_0 R^3}{2r^2} \cos \theta\right\} r^2 \sin \theta d\theta d\phi.$$

The integration with respect to ϕ is clearly equivalent to multiplication by 2π. We have therefore

$$T_f = \frac{\varrho \pi R^6 c_0{}^2}{2r^3}\Bigg|_{r=R} \int_0^\pi \cos^2 \theta \sin \theta d\theta - \frac{\varrho \pi R^6 c_0{}^2}{2r^3}\Bigg|_{r=\infty} \int_0^\pi \cos^2 \theta \sin \theta d\theta.$$

The second integral is obviously equal to zero, and we have

$$T_f = \frac{\varrho \pi R^3 c_0{}^2}{3}.$$

Writing $$m = \frac{4\pi R^3}{3}\varrho,$$

we find $$T_f = \tfrac{1}{4}mc_0{}^2,$$

where m evidently means the mass of the fluid filling a volume equal to that of the sphere of radius R. If M be the mass of the sphere itself, the total kinetic energy of the moving sphere and fluid will be

$$T = \tfrac{1}{2}Mc_0{}^2 + \tfrac{1}{4}mc_0{}^2$$
or $$T = \tfrac{1}{2}(M + \tfrac{1}{2}m)c_0{}^2 \quad . \quad . \quad . \quad (11\cdot25)$$

Briefly we may say that the presence of the fluid has the same effect on the motion of the sphere as if its proper mass were increased by an amount equal to one-half the mass of the fluid filling a volume equal to that of the sphere.

§ 11·3. Waves in Deep Water

Let us think of the liquid as resting on a horizontal surface coincident with the XY plane, the Z axis being directed upwards. We shall inquire about the velocity of harmonic waves (if such waves be possible) travelling in the X direction. Differential quotients with respect to the Y direction are therefore zero and Laplace's equation becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \ldots \ldots \quad (11\cdot3)$$

A suitable equation for such a wave is

$$\phi = A \cos \alpha(x - at) \quad \ldots \ldots \quad (11\cdot31)$$

where α $(= 2\pi/\lambda$, see § 9$)$ and a are constants, and A depends on z only. Substituting in Laplace's equation, we find

$$- \alpha^2 A \cos \alpha(x - at) + \frac{\partial^2 A}{\partial z^2} \cos \alpha(x - at) = 0,$$

and consequently

$$\frac{d^2 A}{dz^2} = \alpha^2 A \ ;$$

from which we derive

$$A = A_0 e^{\alpha z} + B_0 e^{-\alpha z} \quad \ldots \ldots \quad (11\cdot311)$$

A_0 and B_0 being constants. So that $(11\cdot31)$ now becomes

$$\phi = (A_0 e^{\alpha z} + B_0 e^{-\alpha z}) \cos \alpha(x - at) \quad . \quad (11\cdot312)$$

The vertical component of the velocity of the water must be zero at the plane on which it rests, i.e.

$$\left(\frac{\partial \phi}{\partial z} \right)_{z=0} = 0.$$

Therefore

$$\alpha(A_0 e^{\alpha z} - B_0 e^{-\alpha z}) \cos \alpha(x - at)$$

must be zero when $z = 0$. It follows consequently that $A_0 = B_0$ and the expression for ϕ becomes

$$\phi = A_0 (e^{\alpha z} + e^{-\alpha z}) \cos \alpha(x - at) \quad . \quad (11\cdot32)$$

It is convenient to make use of the hydrodynamical equations in the form $(10\cdot9)$, remembering however that $\omega_x = \omega_y = \omega_z = 0$ and replacing u, v and w by the corresponding gradients of the velocity potential. These equations are therefore

$$- \frac{\partial}{\partial x}(\Pi + V + \tfrac{1}{2}c^2) = - \frac{\partial^2 \phi}{\partial t \partial x},$$

$$- \frac{\partial}{\partial y}(\Pi + V + \tfrac{1}{2}c^2) = - \frac{\partial^2 \phi}{\partial t \partial y},$$

$$- \frac{\partial}{\partial z}(\Pi + V + \tfrac{1}{2}c^2) = - \frac{\partial^2 \phi}{\partial t \partial z} \quad . \quad . \quad (11\cdot33)$$

Multiplying by dx, dy and dz respectively and adding we get

$$- d(\Pi + V + \tfrac{1}{2}\mathbf{c}^2) = - d\Big(\frac{\partial \phi}{\partial t}\Big),$$

and therefore

$$\Pi + V + \tfrac{1}{2}\mathbf{c}^2 = \frac{\partial \phi}{\partial t} + C.$$

Replacing Π by $\dfrac{p}{\varrho}$ and V by the gravitational potential, gz, we obtain

$$\frac{p}{\varrho} + gz + \tfrac{1}{2}\mathbf{c}^2 = \frac{\partial \phi}{\partial t} + C \quad . \quad . \quad . \quad \textbf{(11·331)}$$

It will be noted that the process of integration out of which C has arisen leaves open the two possibilities, namely that C is a constant or depends on the time only.

We may now introduce certain approximations, if we agree that the waves are to be restricted to small amplitudes and velocities. On the principle of neglecting squares and products of small quantities we shall cut out the term $\tfrac{1}{2}\mathbf{c}^2$ in $\textbf{(11·331)}$, and since the pressure on the surface of the liquid must be everywhere constant, we have there

$$\frac{Dp}{Dt} = 0.$$

In consequence we get from $\textbf{(11·331)}$

$$g\frac{Dz}{Dt} = \frac{D}{Dt}\Big(\frac{\partial \phi}{\partial t}\Big) + \frac{DC}{Dt} \quad . \quad . \quad . \quad \textbf{(11·332)}$$

This must hold at any rate in the immediate neighbourhood of the liquid surface.[1] Now

$$\frac{Dz}{Dt} = w = - \frac{\partial \phi}{\partial z}$$

and

$$\frac{D}{Dt}\Big(\frac{\partial \phi}{\partial t}\Big) = \frac{\partial^2 \phi}{\partial t^2} + u\frac{\partial}{\partial x}\Big(\frac{\partial \phi}{\partial t}\Big) + v\frac{\partial}{\partial y}\Big(\frac{\partial \phi}{\partial t}\Big) + w\frac{\partial}{\partial z}\Big(\frac{\partial \phi}{\partial t}\Big)$$

which becomes, in consequence of small amplitudes and velocities,

$$\frac{D}{Dt}\Big(\frac{\partial \phi}{\partial t}\Big) = \frac{\partial^2 \phi}{\partial t^2}.$$

Equation $\textbf{(11·332)}$ thus takes the form

$$- g\frac{\partial \phi}{\partial z} = \frac{\partial^2 \phi}{\partial t^2} + \frac{DC}{Dt}.$$

[1] Consequently in the equations which follow z means the whole depth of the water.

If we now substitute in it the expression (**11·32**) for ϕ we arrive at

$$- agA_0(e^{az} - e^{-az}) \cos \alpha(x - at)$$

$$+ \alpha^2 a^2 A_0(e^{az} + e^{-az}) \cos \alpha(x - at) = \frac{dC}{dt},$$

since C is either constant or depends on t only. Or more shortly

$$\frac{dC}{dt} = K \cos \alpha(x - at).$$

The last equation indicates that dC/dt is zero, since if it varied with t it would necessarily vary with x also, and this latter possibility has already been excluded. Hence

$$K = \alpha^2 a^2 A_0(e^{az} + e^{-az}) - agA_0(e^{az} - e^{-az}) = 0,$$

and

$$a^2 = \frac{g}{\alpha} \frac{e^{az} - e^{-az}}{e^{az} + e^{-az}}$$

or

$$a = \sqrt{\frac{g\lambda}{2\pi} \left| \frac{\left(e^{\frac{2\pi}{\lambda}z} - e^{-\frac{2\pi}{\lambda}z}\right)}{\left(e^{\frac{2\pi}{\lambda}z} + e^{-\frac{2\pi}{\lambda}z}\right)} \right|} \quad . \quad . \quad (\mathbf{11·34})$$

In deep water therefore, where $\dfrac{z}{\lambda}$ is very great, the expression for the velocity approximates to

$$a = \sqrt{\frac{g\lambda}{2\pi}} \quad . \quad . \quad . \quad . \quad (\mathbf{11·35})$$

§ 11·4. VORTEX MOTION

A line drawn in the fluid, so that its tangent at every point on it at a given instant coincides with the direction of **ω** at that point, is called a **vortex line**. The equations of a vortex line will therefore be

$$\frac{\delta x}{\omega_x} = \frac{\delta y}{\omega_y} = \frac{\delta z}{\omega_z} \quad . \quad . \quad . \quad . \quad (\mathbf{11·4})$$

if δx, δy and δz represent a short arc measured in the direction of **ω**. If a vortex line be drawn through every point on a closed loop the resulting set of lines constitutes a **vortex tube** and it will be convenient to use the term **vortex filament** for a vortex tube of very small or infinitesimal cross-sectional dimensions.

The following simple example will serve as an illustration. Imagine a fluid in revolution about an axis passing through the origin and suppose the angular velocity $\dot{q}$ of any particle about the axis to be a function of its perpendicular distance from it.

We have for **c** the velocity of a fluid particle (see for example equations **9·61**),

$$\mathbf{c} = [\dot{\mathbf{q}}\mathbf{r}],$$

where **r** is the vectorial distance of the particle from the origin. Therefore

$$u = \dot{q}_y z - \dot{q}_z y,$$
$$v = \dot{q}_z x - \dot{q}_x z,$$
$$w = \dot{q}_x y - \dot{q}_y x.$$

Now suppose the axis of revolution to coincide with the Z axis. The last equations become

$$u = - \dot{q}_z y,$$
$$v = \dot{q}_z x,$$
$$w = 0.$$

Hence

$$2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\dot{q}_z + x\frac{\partial \dot{q}_z}{\partial x} + y\frac{\partial \dot{q}_z}{\partial y},$$

or

$$2\omega_z = 2\dot{q}_z + \frac{x^2}{\varrho}\frac{\partial \dot{q}}{\partial \varrho} + \frac{y^2}{\varrho}\frac{\partial \dot{q}_z}{\partial \varrho},$$

if $\varrho^2 = x^2 + y^2$. Therefore

$$2\omega_z = 2\dot{q}_z + \varrho\frac{\partial \dot{q}_z}{\partial \varrho}$$

or

$$2\boldsymbol{\omega} = 2\dot{\mathbf{q}} + \varrho\frac{\partial \dot{\mathbf{q}}}{\partial \varrho} \qquad . \quad . \quad . \quad . \quad (11\cdot41)$$

which becomes, if $\boldsymbol{\omega}$ is constant,

$$\dot{\mathbf{q}} = \boldsymbol{\omega} + \frac{A}{\varrho^2} \qquad . \quad . \quad . \quad . \quad (11\cdot411)$$

A being a constant of integration. Let the constant rotation $\boldsymbol{\omega}$ be equal to $\boldsymbol{\omega}_0$ when $\varrho \lessgtr \varrho_0$, and zero when $\varrho > \varrho_0$. When therefore $\varrho < \varrho_0$ the constant A must be zero, otherwise $\dot{\mathbf{q}}$ would be infinite in the axis, where $\varrho = 0$. Consequently

$$\dot{\mathbf{q}} = \boldsymbol{\omega}_0 \qquad . \quad . \quad . \quad . \quad . \quad (11\cdot412)$$

within the cylinder of radius ϱ_0 and the fluid within the cylinder will turn about Z like a rigid body. Outside this cylinder $\boldsymbol{\omega} = 0$ and therefore

$$\dot{\mathbf{q}} = \frac{A}{\varrho^2},$$

and if we wish to avoid velocity discontinuities we must have, when $\varrho = \varrho_0$

$$\mathbf{q}_0 = \boldsymbol{\omega}_0 = \frac{A}{\varrho_0{}^2}$$

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Hence outside the cylinder
$$A = \varrho_0{}^2 \boldsymbol{\omega}_0$$
and
$$\mathbf{q} = \frac{\varrho_0{}^2 \boldsymbol{\omega}_0}{\varrho^2} \quad . \quad . \quad . \quad (11\text{·}413)$$

The vortical region is within the cylinder of radius ϱ_0. It is only here that the *rotation* $\boldsymbol{\omega}$ is different from zero. Outside the cylinder it is true that the fluid is travelling round Z, but there is no rotation in the sense in which we are employing the term. The distinction between the two types of motion is illustrated by Fig. 10·8, (a) showing the irrotational motion and (b) the rotational motion. The irrotational motion round the Z axis is like that of a man who walks round a tree while all the time facing north, whereas he would exemplify the rotational type of motion if he were to face steadily in the direction in which he is travelling while going round the tree.

With the help of one or another of the hydrodynamical equations given above we can easily deduce some interesting properties of vortices. Starting with the equation (10·721) in which it will be remembered δx_0, δy_0 and δz_0 represent the vectorial separation of two neighbouring fluid particles at the same instant, t_0, while δx, δy and δz represent the separation of the same particles at some later instant, t; we integrate round a closed loop and thus get

$$\oint (u_0 \delta x_0 + v_0 \delta y_0 + w_0 \delta z_0) = \oint (u\delta x + v\delta y + w\partial z) \quad . \quad (11\text{·}42)$$

since the integral $\oint \left(\dfrac{\partial \chi}{\partial x_0}\delta x_0 + \dfrac{\partial \chi}{\partial y_0}\delta y_0 + \dfrac{\partial \chi}{\partial z_0}\delta z_0 \right)$ obviously vanishes.

Either integral is called the **circulation** round the loop over which it is extended, and since both of the loops thread together the same chain of fluid particles the theorem (**11·42**) affirms that the circulation round a loop connecting a chain of fluid particles remains unchanged in the course of the motion of the particles. The theorem of Stokes enables us to express (**11·42**) in still another way, namely

$$\iint (\mathbf{curl}\ \mathbf{c}_0, \mathbf{dS}_0) = \iint (\mathbf{curl}\ \mathbf{c}, \mathbf{dS}),$$
or, as
$$\mathbf{curl}\ \mathbf{c} = 2\boldsymbol{\omega},$$
$$\iint (\omega_0 dS_0) = \iint (\omega dS) \quad . \quad . \quad (11\text{·}421)$$

This means that the integral $\iint (\boldsymbol{\omega}\mathbf{dS})$ extended over a surface will remain unchanged as the surface is carried along by the motion of the fluid.

The statement (**11·42**) includes the special case that if the circulation, at some instant, taken round any closed loop whatever, is zero, then it will always be zero, and this means that a velocity potential exists (§ 11). It follows from (**11·421**) that if $\boldsymbol{\omega}$ is zero in any portion of the fluid at any instant, it must always be zero in that portion of the fluid.

Since $\boldsymbol{\omega}$ is the **curl** of a vector ($\boldsymbol{\omega} = \frac{1}{2}$ **curl c**), div $\boldsymbol{\omega}$ must be zero, by equation (**2·42**). Therefore the integral

$$\iiint \mathbf{div}\ \boldsymbol{\omega}\ dxdydz,$$

extended over the fluid contained at a given instant within a closed surface must be zero also, and by the theorem of Gauss

$$\iint (\boldsymbol{\omega}\ \mathbf{dS}) = 0 \quad . \quad . \quad . \quad . \quad (11\text{·}43)$$

when the surface integral is extended over the closed surface and **dS** has the direction of the outward normal. If now the
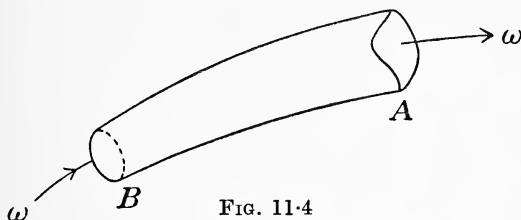
Fɪɢ. 11·4

closed surface be part of a vortex tube bounded by two cross-sectional surfaces A and B (Fig. 11·4), the part of (**11·43**) which extends *round* the tube must be zero because $\boldsymbol{\omega}$ is parallel to the side of the tube, and therefore perpendicular to the direction of the vector **dS**. We are consequently left with the surface integral over the cross-sectional faces A and B, and so

$$\iint_A (\boldsymbol{\omega}\ \mathbf{dS}) + \iint_B (\boldsymbol{\omega}\ \mathbf{dS}) = 0 \quad . \quad (11\text{·}431)$$

The vector **dS**, having in (**11·431**) the direction of the outward normal, will have its vectorial arrow passing through A in the same sense as that of $\boldsymbol{\omega}$. At the surface B the two directions have opposite senses. If we agree to reverse the sense of **dS** on the surface B, so that the vectorial arrows of $\boldsymbol{\omega}$ and **dS** cross both surfaces A and B *in the same sense*, equation (**11·431**) becomes

$$\iint_A (\boldsymbol{\omega}\ \mathbf{dS}) = \iint_B (\boldsymbol{\omega}\ \mathbf{dS}) \quad . \quad . \quad (11\text{·}432)$$

We shall call $\boldsymbol{\omega}$ the **vortex intensity** and the integral $\iint\limits_{A} (\boldsymbol{\omega}\ \mathbf{dS})$ the **vortex flux** across the area A. We have thus learned two things about the vortex flux ; firstly that the flux across an area A remains unchanged as it is carried along by the motion of the fluid (**11·421**) and, secondly, that the flux through A is equal to the flux through any other section of the same vortex tube (**11·432**). The method by which (**11·432**) was established clearly demonstrates that a vortex tube must either extend to the boundaries of the fluid or, failing that, it must run into itself and constitute a **vortex ring.** There is one other feature of vortices which the same method demonstrates. Consider any surface made up of vortex lines. It may constitute a sort of longitudinal section of a vortex, or it may be a surface enclosing a vortex tube. In either case the integral $\iint (\boldsymbol{\omega}\ \mathbf{dS})$ over the surface, or over any part of it, must be zero, since in the surface the vectors $\boldsymbol{\omega}$ and $\mathbf{dS}$ are perpendicular to one another ; and it will remain zero as the surface is carried along by the motion of the fluid. If two such surfaces intersect, they must do so in a vortex line, and consequently they will continue to intersect in a vortex line as they are carried along by the motion of the fluid. In other words, if a chain of fluid particles lies along a vortex line at any instant it will always lie on a vortex line. Consequently too the particles which are on the boundary of a vortex at any instant will always continue on its boundary. Vortices have therefore a quality of permanence. They cannot be created or destroyed. It must be remembered, however, in connexion with this last statement, that we have assumed no viscosity and also that the body force in the fluid has a potential. These assumptions however are not necessary for the validity of (**11·432**) which depends on the fact that the divergence of a curl is identically zero.

The analogy between the lamellar (i.e. potential) flow in an incompressible fluid and an electrostatic field free from charges has already been pointed out. There is also a close analogy between the rotational fluid motion we have just been studying and the magnetic field due to a current in a wire. The analogy is very close indeed when the fluid is incompressible. The flux in a vortex or vortex ring is analogous to the current, the vortex intensity corresponding to current density, while $\mathbf{c} \equiv (u,\ v,\ w)$ corresponds to the magnetic field intensity. If suitable units for current and field intensity are used the correspondence is exact.

CHAPTER IX

MOTION IN VISCOUS FLUIDS

§ 11·5. Equations of Motion in a Viscous Fluid

WE shall use the equations (10·2) as a starting point for developing those of a viscous medium. As explained in § 10·5, we may replace $\varrho\dfrac{\partial^2\alpha}{\partial t^2}$, $\varrho\dfrac{\partial^2\beta}{\partial t^2}$ and $\varrho\dfrac{\partial^2\gamma}{\partial t^2}$ by $\varrho\dfrac{Du}{Dt}$, $\varrho\dfrac{Dv}{\partial t}$ and $\varrho\dfrac{Dw}{Dt}$ respectively; and if we further suppose the body force to have a potential, so that $R_x = -\varrho\dfrac{\partial V}{\partial x}$, for example, the equations will assume the form

$$\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} - \varrho\frac{\partial V}{\partial x} = \varrho\frac{Du}{Dt},$$

$$\frac{\partial t_{yx}}{\partial x} + \frac{\partial t_{yy}}{\partial y} + \frac{\partial t_{yz}}{\partial z} - \varrho\frac{\partial V}{\partial y} = \varrho\frac{Dv}{Dt}$$

$$\frac{\partial t_{zx}}{\partial x} + \frac{\partial t_{zy}}{\partial y} + \frac{\partial t_{zz}}{\partial z} - \varrho\frac{\partial V}{\partial z} = \varrho\frac{Dw}{Dt}. \quad . \quad . \quad . \quad \textbf{(11·5)}$$

The components t_{xx}, t_{xy}, etc., of the stress tensor now include additional terms due to the friction between one part of the medium and another. We must therefore write

$$t_{xx} = t'_{xx} + t''_{xx},$$
$$t_{xy} = t'_{xy} + t''_{xy},$$

and corresponding equations for the remaining components. In these equations t'_{xx}, t'_{xy}, etc., mean the part of the stress associated with strain, i.e. the elastic part of the stress; while t''_{xx}, t''_{xy}, etc., represent the part of the tensor evoked by the friction between the parts of the medium. In a fluid medium, to which we now confine our attention,

$$t'_{xx} = t'_{yy} = t'_{zz} = -p,$$

where p is the pressure and t'_{xy}, t'_{xz}, t'_{yz}, etc., are all zero. Now the part of the stress tensor due to viscosity or internal friction,

namely t''_{xx}, t''_{xy}, t''_{yz}, etc., is related to the velocity gradients $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial z}$, etc., in precisely the same way as the elastic part of the tensor, namely t'_{xx}, t'_{xy}, t'_{yz}, etc., is related to the displacement tensor $\dfrac{\partial a}{\partial x}$, $\dfrac{\partial a}{\partial y}$, $\dfrac{\partial \beta}{\partial z}$, etc. Therefore we have

$$t''_{xx} = \left(k' + \frac{4n'}{3}\right)\frac{\partial u}{\partial x} + \left(k' - \frac{2n'}{3}\right)\frac{\partial v}{\partial y} + \left(k' - \frac{2n'}{3}\right)\frac{\partial w}{\partial z},$$

$$t''_{yy} = \left(k' - \frac{2n'}{3}\right)\frac{\partial u}{\partial x} + \left(k' + \frac{4n'}{3}\right)\frac{\partial v}{\partial y} + \left(k' - \frac{2n'}{3}\right)\frac{\partial w}{\partial z},$$

$$t''_{zz} = \left(k' - \frac{2n'}{3}\right)\frac{\partial u}{\partial x} + \left(k' - \frac{2n'}{3}\right)\frac{\partial v}{\partial y} + \left(k' + \frac{4n'}{3}\right)\frac{\partial w}{\partial z}.$$

$$t''_{xy} = n'\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \textbf{(11·51)}$$

and so on, (**10·01** and **10·02**) k' and n' being constants. We obtain consequently for a viscous fluid,

$$-\frac{\partial p}{\partial x} + n'\nabla^2 u + \left(k' + \frac{n'}{3}\right)\frac{\partial}{\partial x}(\text{div } \mathbf{c}) - \varrho\frac{\partial V}{\partial x} = \varrho\frac{Du}{Dt},$$

$$-\frac{\partial p}{\partial y} + n'\nabla^2 v + \left(k' + \frac{n'}{3}\right)\frac{\partial}{\partial y}(\text{div } \mathbf{c}) - \varrho\frac{\partial V}{\partial y} = \varrho\frac{Dv}{Dt},$$

$$-\frac{\partial p}{\partial z} + n'\nabla^2 w + \left(k' + \frac{n'}{3}\right)\frac{\partial}{\partial z}(\text{div } \mathbf{c}) - \varrho\frac{\partial V}{\partial z} = \varrho\frac{Dw}{Dt}. \quad \textbf{(11·52)}$$

We shall assume k' to be zero, and replace n' by μ so that we get

$$-\frac{\partial p}{\partial x} + \mu\nabla^2 u + \frac{\mu}{3}\frac{\partial}{\partial x}(\text{div } \mathbf{c}) - \varrho\frac{\partial V}{\partial x} = \varrho\frac{Du}{Dt} \quad . \quad . \quad \textbf{(11·521)}$$

and two similar equations for $\dfrac{Dv}{Dt}$ and $\dfrac{Dw}{Dt}$. If the medium is incompressible **div c** $= 0$ and these equations become :

$$-\frac{\partial p}{\partial x} + \mu\nabla^2 u - \varrho\frac{\partial V}{\partial x} = \varrho\frac{Du}{Dt},$$

$$-\frac{\partial p}{\partial y} + \mu\nabla^2 v - \varrho\frac{\partial V}{\partial y} = \varrho\frac{Dv}{Dt},$$

$$-\frac{\partial p}{\partial z} + \mu\nabla^2 w - \varrho\frac{\partial V}{\partial z} = \varrho\frac{Dw}{Dt} \quad . \quad . \quad \textbf{(11·522)}$$

The constant μ is called the **coefficient of viscosity,** or briefly the **viscosity** of the fluid.

§ 11·6. Poiseuille's Formula

We shall now apply these equations to the problem of the steady flow of a liquid (incompressible fluid) along a horizontal tube of small radius, R. The axis of the tube may be taken to coincide with the Z axis of rectangular co-ordinates, the direction of flow being that of the axis, and we may drop the potential terms $\varrho \dfrac{\partial V}{\partial x}$, etc. In addition to the equations of motion (11·522), the following conditions have to be satisfied:

$$u = v = 0,$$

$$\operatorname{\mathbf{div}} \mathbf{c} = \frac{\partial w}{\partial z} = 0,$$

$$w = \text{function } (r)$$

$$\frac{\partial \psi}{\partial t} = 0,$$

$$\mathbf{c} = 0 \text{ when } r = R. \quad . \quad . \quad (11·6)$$

In these equations, r is the perpendicular distance of any point in the fluid from the axis, and ψ may mean any quantity associated with the motion of the fluid. The last of the statements (11·6) affirms that the liquid is at rest at the wall of the tube. Experiment indicates that this is at all events very near the truth.

In consequence of the conditions (11·6) the equations of motion become

$$-\frac{\partial p}{\partial x} = 0,$$

$$-\frac{\partial p}{\partial y} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11·61)$$

$$-\frac{\partial p}{\partial z} + \mu \nabla^2 w = 0.$$

It follows at once that p is constant over any cross-section of the tube, and is consequently a function of z only; while

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}.$$

Now

$$\frac{\partial w}{\partial x} = \frac{x}{r} \frac{\partial w}{\partial r},$$

since w is a function of the single variable, r, and hence

$$\frac{\partial^2 w}{\partial x^2} = \frac{x^2}{r^2} \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} - \frac{x^2}{r^3} \frac{\partial w}{\partial r},$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{y^2}{r^2} \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} - \frac{y^2}{r^3} \frac{\partial w}{\partial r},$$

and, on adding, since $x^2 + y^2 = r^2$,

$$\nabla^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}$$

or, finally

$$\nabla^2 w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) \quad . \quad . \quad . \quad \textbf{(11·611)}$$

Thus the last of the equations **(11·61)** becomes

$$\frac{dp}{dz} = \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) \quad . \quad . \quad . \quad . \quad \textbf{(11·62)}$$

the straight d of ordinary differentiation having been introduced to mark the fact that the left-hand member of the equation is a function of z only, and the right-hand member a function of r only.   It follows that

$$\frac{dp}{dz} = G,$$

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = G \quad . \quad . \quad . \quad \textbf{(11·621)}$$

where G is a constant.
    From the second of these equations we get

$$\mu\frac{d}{dr}\left(r\frac{dw}{dr}\right) = Gr,$$

and consequently

$$\mu r\frac{dw}{dr} = \frac{Gr^2}{2} + A,$$

where A is a constant of integration.   This holds for all values of r from zero to R, and on substituting the particular value 0, we see that A must be zero.   The equation therefore becomes

$$\mu r\frac{dw}{dr} = \frac{Gr^2}{2},$$

or

$$\mu\frac{dw}{dr} = \frac{Gr}{2} ;$$

which on integration gives us

$$\mu w = \frac{Gr^2}{4} + B.$$

Since by hypothesis $w = 0$ at the wall of the tube, we must have

$$0 = \frac{GR^2}{4} + B,$$

and therefore, on subtraction,

$$w = \frac{G}{4\mu}(r^2 - R^2) \quad . \quad . \quad . \quad . \quad \textbf{(11·63)}$$

We shall now deduce an expression for the volume of liquid passing any cross-section of the tube per second. The area of the part of the cross-section bounded by the circles of radii r and $r + dr$ is

$$2\pi r dr$$

and the volume of liquid passing per second through this is

$$2\pi w r dr,$$

w being the velocity at the distance r from the axis. Hence if Q be the volume flowing through the whole cross-section per second,

$$Q = 2\pi \int_0^R wr dr.$$

Consequently

$$Q = 2\pi \int_0^R \frac{G}{4\mu}(r^3 - R^2 r)dr,$$

or

$$Q = -\frac{G\pi R^4}{8\mu} \quad . \quad . \quad . \quad . \quad . \quad (11 \cdot 64)$$

From the first of the equations $(11 \cdot 621)$ we have

$$G = \frac{p_2 - p_1}{l} \quad . \quad . \quad . \quad (11 \cdot 641)$$

where p_1 and p_2 are the pressures at two cross-sections separated by the distance l, the liquid flowing from 1 to 2. On substituting in $(11 \cdot 64)$ we obtain the well-known formula of Poiseuille

$$Q = \frac{(p_1 - p_2)\pi R^4}{8\mu l} \quad . \quad . \quad . \quad (11 \cdot 65)$$

The foregoing theory of the flow of a viscous liquid through a narrow tube constitutes the basis of a method of measuring the coefficients of viscosity—or the viscosities, as we say for brevity—of liquids.

If the tube is not horizontal we shall have in place of the gradient

$$G = \frac{\partial p}{\partial z}$$

of $(11 \cdot 621)$ another constant, namely

$$G = \frac{\partial}{\partial z}(p + \varrho V) . \quad . \quad . \quad . \quad (11 \cdot 66)$$

as is evident from equations $(11 \cdot 522)$. If for example the tube be vertical, and the liquid flowing down it (Z axis directed

downwards), the gravitational force per unit mass is g and hence $V = -gz + \text{constant}$. We may as well take the constant to be zero—V only appearing in $\dfrac{\partial V}{\partial z}$—and we find

$$G = \frac{\partial}{\partial z}(p - \varrho gz),$$

so that instead of (**11·65**) we shall have

$$Q = \frac{\{p_1 - p_2 + \varrho gl\}\pi R^4}{8\mu l} \qquad . \quad . \quad (11\cdot661)$$

If the apparatus be arranged—as it sometimes is—after the manner illustrated in Fig. 11·6, it is not permissible to take the pressure difference between such a point as A and the point B at the end of the tube as equivalent to $p_1 - p_2$. If the velocity is practically zero at A, we have to subtract from the pressure at A the amount
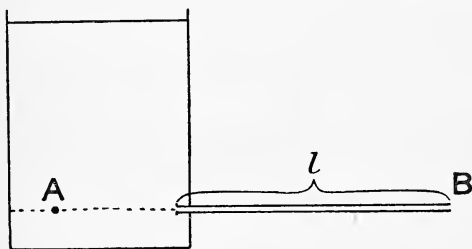
Fig. 11·6

$\frac{1}{2}\varrho c^2$ to get p_1, the pressure just inside the tube, in accordance with Bernoulli's theorem.

The formula (**11·65**) applies, as we have seen, to the case of an incompressible fluid ; in practice it applies to liquids. We can however very easily modify it to obtain a formula applicable to gases—or, to be precise, to fluids obeying Boyle's law. The gradient $\dfrac{\partial w}{\partial z}$ is no longer zero and consequently w is a function of r and z. Apart from this the conditions (**11·6**) continue in force. Instead of (**11·62**) we now find

$$\frac{\partial p}{\partial z} = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{4}{3}\mu\frac{\partial^2 w}{\partial z^2},$$

since $\nabla^2 w$ must include the term $\dfrac{\partial^2 w}{\partial z^2}$, and we have the term $\dfrac{\mu}{3}\dfrac{\partial}{\partial z}$ (**div c**) of (**11·521**). In consequence of Boyle's law w will vary inversely as the pressure at all points at the same distance from the axis. Hence if the pressure gradient is everywhere small, $\dfrac{\partial w}{\partial z}$ will also be small and $\dfrac{\partial^2 w}{\partial z^2}$ negligible. We may there-

fore adopt the equations (**11·621**), provided of course we do not lose sight of the fact that

$$G = \frac{dp}{dz}$$

is no longer a constant, but a function of z, that is to say it varies from one cross-section to another. The volume—call it W—flowing per second through the cross-section at z will therefore be

$$W = -\frac{dp}{dz}\frac{\pi R^4}{8\mu},$$

where $\frac{dp}{dz}$ has the value appropriate to that particular cross-section. On multiplying both sides by p, the pressure, we get

$$pW = -\frac{p\,dp}{dz}\frac{\pi R^4}{8\mu}.$$

Now, in consequence of Boyle's law pW has the same value for all cross-sections. Let us call it Q. It represents the quantity of gas passing through the tube, or past any cross-section, per second; the unit used being that quantity of the gas for which the product of pressure and volume is unity. Therefore

$$Q = -\frac{d(p^2)}{dz}\frac{\pi R^4}{16\mu},$$

or
$$Q = \frac{(p_1{}^2 - p_2{}^2)\pi R^4}{16\mu l} \qquad . \quad . \quad . \quad . \quad (11\cdot67)$$

§ 11·7. MOTION OF A SPHERE THROUGH A VISCOUS LIQUID. FORMULA OF STOKES

The special problem to which we now give our attention is that of determining the force required to keep a sphere in motion with a constant velocity, through an infinitely extended mass of liquid (incompressible fluid). We shall represent the constant velocity by c_0 and suppose, in the first instance, the centre of the sphere to be travelling along the Z axis in the positive direction. The problem is equivalent to that which arises if we imagine a uniform velocity $\equiv (0, 0, -c_0)$ superposed on the whole system of fluid and sphere. So that the sphere is now at rest—and we shall suppose its centre to be at the origin of our system of co-ordinates—and the infinitely distant parts of the liquid have the velocity $(0, 0, -c_0)$. Obviously we may take the pressure to be constant at distant points and it will be immaterial what value we assign to this constant. It is

convenient to take it to be zero. We are not concerned with any body forces and the equations of motion of the liquid (**11·522**) become

$$-\frac{\partial p}{\partial x} + \mu \nabla^2 u = \varrho \frac{Du}{Dt},$$

$$-\frac{\partial p}{\partial y} + \mu \nabla^2 v = \varrho \frac{Dv}{Dt},$$

$$-\frac{\partial p}{\partial z} + \mu \nabla^2 w = \varrho \frac{Dw}{Dt}. \quad . \quad . \quad . \quad (11·7)$$

Let R be the radius of the sphere. The conditions to be satisfied are the following :

$u = v = w = 0$, when $r = R$,

div c $= 0$,

$\mathbf{c} \equiv (0, 0, -c_0)$, when $r = \infty$ . . . . (**11·71**)

$\frac{\partial \psi}{\partial t} = 0$ for all quantities, ψ, associated with the motion ;

and we shall impose the restriction that u, v and w are every-where small. This last condition justifies us in ignoring $\frac{Du}{Dt}, \frac{Dv}{Dt}$ and $\frac{Dw}{Dt}$. In fact

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$

or $\frac{Du}{Dt} = 0 + $ sum of products of small quantities taken two at a time.

The ignoration of the accelerations simplifies the equations of motion to

$$\frac{\partial p}{\partial x} = \mu \nabla^2 u,$$

$$\frac{\partial p}{\partial y} = \mu \nabla^2 v,$$

$$\frac{\partial p}{\partial z} = \mu \nabla^2 w. \quad . \quad . \quad . \quad . \quad (11·72)$$

By differentiating with respect to x, y and z respectively and adding, we obtain

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \mu \nabla^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right),$$

or $\qquad\qquad \nabla^2 p = \mu \nabla^2 (\mathbf{div\ c})$

Consequently $\qquad\qquad \nabla^2 p = 0 \quad . \quad . \quad . \quad . \quad (11·721)$

We naturally think at first of a velocity potential; but a little reflexion will show that a velocity potential cannot exist. Consider, for example, the state of affairs at a point on the x axis close to the spherical surface. Here quite obviously

$$\frac{\partial w}{\partial x} \neq 0,$$

while

$$\frac{\partial u}{\partial z} = 0,$$

and hence

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

differs from zero at such a point; and this is incompatible with the existence of a velocity potential. The kind of symmetry which the motion possesses leads us to suspect that it is the Z component of the velocity, w, which makes a velocity potential an impossibility, and we endeavour therefore to satisfy the conditions of the problem by

$$u = -\frac{\partial \phi}{\partial x},$$

$$v = -\frac{\partial \phi}{\partial y},$$

$$w = -\frac{\partial \phi}{\partial z} + w_1. \quad . \quad . \quad . \quad (11\cdot73)$$

Substitution in $(11\cdot72)$ leads to

$$\frac{\partial p}{\partial x} = -\mu \frac{\partial}{\partial x} \nabla^2 \phi,$$

$$\frac{\partial p}{\partial y} = -\mu \frac{\partial}{\partial y} \nabla^2 \phi,$$

$$\frac{\partial p}{\partial z} = -\mu \frac{\partial}{\partial z} \nabla^2 \phi + \mu \nabla^2 w_1 \quad . \quad . \quad (11\cdot731)$$

while substitution in the divergence equation,

$$\mathbf{div\ c} = 0,$$

leads to

$$\frac{\partial w_1}{\partial z} = \nabla^2 \phi \quad . \quad . \quad . \quad . \quad (11\cdot732)$$

We proceed further by adopting the simplest method of satisfying $(11\cdot731)$ and $(11\cdot732)$, namely

$$p = -\mu \nabla^2 \phi,$$
$$\nabla^2 w_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad (11\cdot74)$$

The pressure, p, has to satisfy (**11·721**) and we shall try the solution

$$p = \text{constant} \times \frac{\partial}{\partial z}\left(\frac{1}{r}\right),$$

(see **11·03**), where r is the radial distance from the centre of the sphere. This expression has the sort of axial symmetry characteristic of the motion. Let us write it in the form

$$p = \frac{Az}{r^3} \quad . \quad . \quad . \quad . \quad (\mathbf{11·741})$$

We shall see, as we proceed, that it is the right expression for p if we assign a suitable value to the constant, A. It now follows from equations (**11·732**) and (**11·74**) that

$$w_1 = \frac{A}{\mu r} \quad . \quad . \quad . \quad . \quad (\mathbf{11·742})$$

We reject the possible additive constant, since it may be considered to be included in the constant velocity, $-\mathbf{c}_0$, of the distant parts of the liquid.

Turning to the function ϕ, the problem of § 11·2 suggests putting

$$\phi = az + \frac{bR^3}{2r^3}z + \phi_1 \quad . \quad . \quad . \quad (\mathbf{11·75})$$

The constants a and b, like the constant A above, have still to be determined ; as also has the character of the function ϕ_1. We have therefore

$$\nabla^2\phi = \nabla^2\phi_1,$$

since the first two terms in (**11·75**) contribute zero to $\nabla^2\phi$ (see § 11·2). Consequently

$$p = -\mu\nabla^2\phi_1 \quad . \quad . \quad (\mathbf{11·751})$$

or

$$\frac{Az}{r^3} = -\mu\nabla^2\phi_1.$$

It is easy to verify that this is satisfied by

$$\phi_1 = \frac{Az}{2\mu r} . \quad . \quad . \quad . \quad (\mathbf{11·752})$$

We obtain, in fact, from this expression for ϕ_1

$$\frac{\partial^2\phi_1}{\partial x^2} = -\frac{Az}{2\mu r^3} + \frac{3Azx^2}{2\mu r^5},$$

$$\frac{\partial^2\phi_1}{\partial y^2} = -\frac{Az}{2\mu r^3} + \frac{3Azy^2}{2\mu r^5},$$

and
$$\frac{\partial^2 \phi_1}{\partial z^2} = -\frac{Az}{2\mu r^3} + \frac{3Az^3}{2\mu r^5} - \frac{Az}{\mu r^3},$$

and on adding these together
$$\nabla^2 \phi_1 = -\frac{Az}{\mu r^3},$$

or
$$\nabla^2 \phi_1 = -\frac{p}{\mu},$$

so that (**11·751**) is satisfied.

In virtue of (**11·752**) the expression for ϕ becomes
$$\phi = az + \frac{bR^3 z}{2r^3} + \frac{Az}{2\mu r} \qquad . \quad . \quad . \quad (11\cdot76)$$

and we find for u, v and w (**11·73** and **11·742**)

$$u = \left(\frac{3bR^3}{r^2} + \frac{A}{\mu}\right)\frac{zx}{2r^3},$$

$$v = \left(\frac{3bR^3}{r^2} + \frac{A}{\mu}\right)\frac{zy}{2r^3}, \qquad . \quad . \quad . \quad . \quad (11\cdot761)$$

$$w = \left(\frac{3bR^3}{r^2} + \frac{A}{\mu}\right)\frac{z^2}{2r^3} + \left(\frac{A}{2\mu r} - \frac{bR^3}{2r^3}\right) - a$$

What we have succeeded in doing so far amounts to finding expressions for p, u, v and w which satisfy the equations of motion. We have now to investigate whether we can satisfy the conditions (**11·71**) by assigning suitable values to the constants A, a and b.

Now at infinity $w = -\mathbf{c}_0$ and in consequence $a = \mathbf{c}_0$, and the first two equations (**11·761**) will conform to the condition $u = v = w = 0$ for $r = R$ if
$$A = -3\mu bR,$$

while the last of the equations (**11·761**) will conform to this condition if, in addition to assigning the values just mentioned to a and A, we put $b = -\dfrac{\mathbf{c}_0}{2}$. Therefore

$$A = \frac{3\mu c_0 R}{2},$$

$$a = \mathbf{c}_0$$

$$b = -\frac{\mathbf{c}_0}{2} \quad . \quad . \quad . \quad . \quad (11\cdot762)$$

On substituting these values for A, a and b we get

$$\phi = c_0 z - \frac{c_0 R^3 z}{4r^3} + \frac{3c_0 Rz}{4r} \quad \ldots \ldots \quad (11\cdot 77)$$

$$p = \frac{3\mu c_0 Rz}{2r^3} \quad \ldots \ldots \quad (11\cdot 78)$$

$$u = \left(1 - \frac{R^2}{r^2}\right)\frac{3c_0 Rzx}{4r^3},$$

$$v = \left(1 - \frac{R^2}{r^2}\right)\frac{3c_0 Rzy}{4r^3} \quad \ldots \ldots \quad (11\cdot 781)$$

$$w = \left(1 - \frac{R^2}{r^2}\right)\frac{3c_0 Rz^2}{4r^3} + \frac{c_0 R}{4r}\left(\frac{R^2}{r^2} + 3\right) - c_0.$$

We have thus succeeded in satisfying the equations of motion and the boundary conditions as well.

To get the force exerted by the liquid on the spherical surface, we turn back to $(9\cdot 72)$ which gives us expressions for the force exerted over an element of surface $\mathbf{dS}$. Since the resultant force is obviously along the line of the Z axis, we only need

$$f_z = t_{zx}dS_x + t_{zy}dS_y + t_{zz}dS_z.$$

This will represent the Z component of the force on an element $\mathbf{dS}$ of the surface of the sphere if the vectorial arrow of $\mathbf{dS}$ be directed *away* from the centre of the sphere. Consider in the first place an element $\mathbf{dS}$ in the plane $Y = 0$.

and

$$dS_y = 0$$
$$f_z = t_{zz}dS_z + t_{zx}dS_x,$$

or

$$f_z = (t_{zz}\cos\theta + t_{zx}\sin\theta)\mathbf{dS} \quad \ldots \quad (11\cdot 782)$$

(Fig. $11\cdot 7$). For any other element of area within the zone bounded by the angles θ and $\theta + d\theta$

$$f_z = \Gamma\mathbf{dS},$$

where Γ has the same value as $t_{zz}\cos\theta + t_{zx}\sin\theta$ in $(11\cdot 782)$. We therefore get for $dF = \Sigma f_z$, the Z component of the force exerted on the zonal surface, the expression

$$dF = (t_{zz}\cos\theta + t_{zx}\sin\theta)2\pi R^2 \sin\theta d\theta ;$$

so that

$$F = \int_0^\pi 2\pi R^2 \sin\theta d\theta(t_{zz}\cos\theta + t_{zx}\sin\theta) \quad \ldots \quad (11\cdot 79)$$

Now

$$t_{zz} = -p + \left(k' + \frac{4n'}{3}\right)\frac{\partial w}{\partial z} + \left(k' - \frac{2n'}{3}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right),$$

by § 11·5 ; or

$$t_{zz} = -p + 2n'\frac{\partial w}{\partial z},$$

since **div c** $= 0$; and

$$t_{zx} = n'\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right).$$

Replacing n' by μ we get

$$t_{zz} = -p + 2\mu\frac{\partial w}{\partial z},$$

$$t_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) . \quad . \quad . \quad . \quad (11·791)$$
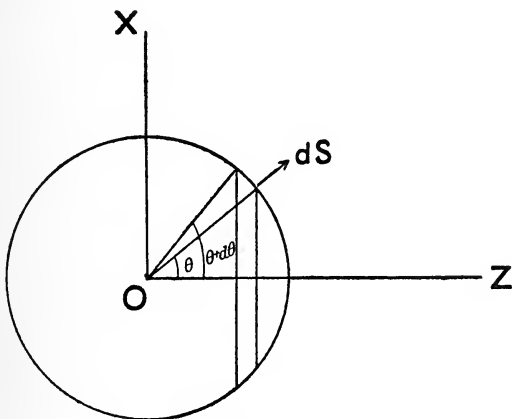
FIG. 11·7

On carrying out the differentiations $\dfrac{\partial w}{\partial z}$, $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial u}{\partial z}$ and substituting the special value R for r, we get, since $\dfrac{z}{R} = \cos\theta$,

$$\left(\frac{\partial w}{\partial z}\right)_{r=R} = -\frac{3c_0}{2R}(\cos\theta \sin^2\theta),$$

$$\left(\frac{\partial w}{\partial x}\right)_{r=R} = -\frac{3c_0}{2R}(\sin^3\theta),$$

$$\left(\frac{\partial u}{\partial z}\right)_{r=R} = \frac{3c_0}{2R}\sin\theta \cos^2\theta.$$

Furthermore we get from (11·78)

$$p = \frac{3\mu c_0 \cos\theta}{2R}.$$

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Thus $\qquad t_{zz} = -\dfrac{3\mu c_0}{2R}\cos\theta - \dfrac{3\mu c_0}{R}\cos\theta\sin^2\theta,$

$$t_{zx} = \frac{3\mu c_0}{2R}(\sin\theta\cos^2\theta - \sin^3\theta).$$

On substituting these expressions for t_{zz} and t_{zx} in (11·79) we get

$$F = -6\pi R\mu c_0 \int_0^{\pi} \sin\theta\, d\theta\{\tfrac{1}{2}\cos^2\theta + \cos^2\theta\sin^2\theta$$

$$-\tfrac{1}{2}\cos^2\theta\sin^2\theta + \tfrac{1}{2}\sin^4\theta\},$$

whence

$$F = -6\pi R\mu \mathbf{c_0} \quad . \quad . \quad . \quad . \quad (11·792)$$

This means that the force is numerically equal to $6\pi R\mu c_0$ and is in the direction in which the liquid is flowing. Finally let us superpose on the whole system the velocity $\mathbf{c_0}$. The liquid will now be at rest at infinitely distant points and the sphere will be travelling in the Z direction with the velocity $\mathbf{c_0}$. It will experience a resisting force equal to $6\pi\mu R\mathbf{c_0}$. This is the celebrated formula of Stokes. In deducing it we have assumed u, v and w to be everywhere small and **div c** to be zero. If u, v and w are everywhere small **div c** will be a small quantity of the second order and may be ignored. This justifies the use of the formula for the slow motion of a sphere through a gas. A further assumption, which has been tacitly made, is that of the continuous character of the medium through which the sphere is moving. The formula begins to be inaccurate, as the experiments of Millikan have shown, when the radius of the sphere approaches in magnitude the mean distance between the molecules or particles of which the medium is constituted.

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CHAPTER X

KINETIC THEORY OF GASES

§ 11·8. Foundations of the Kinetic Theory—
Historical Note

THE kinetic theory (whether applied to gases or to other states) aims at interpreting thermal phenomena in mechanical terms. It assumes that matter in bulk is constituted of innumerable small particles or dynamical systems (molecules) and identifies heat with their kinetic energy.[1] The picture which the theory gives us of a gas is that of an enormously large number of very minute particles flying about in a chaotic manner in the containing vessel, their collisions with the wall of the vessel giving rise to the pressure characteristic of gases. Thermal conductivity and viscosity are explained by the collisions between the individual particles. The velocities of translation account for the laws of diffusion of gases and liquids. The assumption of forces of cohesion between the particles or molecules together with the fact that, though very minute, they have an appreciable proper volume of their own, as distinct from the space they may be said to occupy, renders some account of the liquid and vapour states and the transition from one to the other, as well as of the phenomena of surface tension. The utility of the theory is limited, roughly speaking, to gases; but here it has achieved a great measure of success.

Daniell Bernoulli appears to have been the first to make progress worthy of mention in the development of the theory. He succeeded in accounting for Boyle's law (*Hydrodynamica*, 1738); but nothing further of any consequence was accomplished for about a century, by which time Bernoulli's contribution had been forgotten. In 1845 Waterston submitted a paper on the subject to the Royal Society. Unfortunately there were certain errors in it, and in consequence it was not published at the time. It contained among other things the theorem of equi-

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[1] Strictly speaking, heat is identified with the mechanical energy, kinetic and potential, of the molecules. In the case of a gas however this is practically equivalent to identifying heat and kinetic energy.

partition of energy and an explanation of Avogadro's law, and was eventually published in 1892 because of its historical interest. The further development of the theory is largely due to Clausius (1857) and especially to Clerk Maxwell (1859) and Ludwig Boltzmann (1868). Clerk Maxwell's great contribution was the law of distribution of velocities, while that of Boltzmann consisted in expressing the thermodynamic concept of **entropy** in terms of the probability of the state of an assemblage of molecules or dynamical systems, and this aspect of the theory is the main feature of the great work on statistical mechanics by Willard Gibbs (1901).

§ 11·9. Boyle's Law

We shall begin with the simplest possible assumptions about the constitution of a gas, namely that it consists of minute and perfectly elastic particles, so small that we may regard their proper volume (i.e. $\Sigma \frac{4}{3}\pi r^3$, if they are spheres and r is the radius of a sphere, or $N\frac{4}{3}\pi r^3$, if all have the same radius and N is the total number of them) as a negligible fraction of the volume *occupied* by the gas. Let us further suppose these particles or molecules to be flying about in the containing vessel with very high velocities, so that we may neglect gravity. Their kinetic energy will be maintained by impacts with the wall of the vessel, since we identify it with the heat energy of the gas, and this will be maintained if the temperature of the wall is kept up. We shall provisionally make the further assumption that the wall of the containing vessel is perfectly smooth and elastic, though this is formally inconsistent with the implied hypothesis that the wall of the vessel is itself constituted of molecules. Finally let us suppose that there are no intermolecular forces.
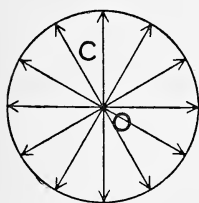
Fig. 11·9

It is not difficult to obtain an expression for the pressure exerted by the gas on the wall of the vessel, assuming it to be due of course to bombardment by gas molecules. It is best to begin by calculating the part of the pressure due to the molecules which have velocities of the same absolute value **c**. These velocities may be supposed to be uniformly distributed as regards directions. This means that, if we draw a line to represent in magnitude and direction the velocity of every molecule in the unit volume, all the lines being drawn from the same point, O (Fig. 11·9), their extremities will be uniformly distributed over a sphere of radius **c**. It is to be understood that an element

of the surface of this sphere, notwithstanding its minuteness, nevertheless embraces an enormous number of these terminal points. If n be the number of the molecules per unit volume, the number of them whose representative points lie on the unit area of the sphere will be

$$\frac{n}{4\pi\mathbf{c}^2}$$

and the number on a small area da will be

$$\frac{nda}{4\pi\mathbf{c}^2}.$$

But $da/\mathbf{c}^2$ is the solid angle subtended at the centre of the sphere by da, and consequently the number of molecules whose directions of motion lie within the limits of a solid angle, $d\Omega$, will be

$$n' = \frac{nd\Omega}{4\pi} \quad . \quad . \quad . \quad . \quad . \quad (11\cdot9)$$

If we take for $d\Omega$ the solid angle contained between the polar angles θ, $\theta + d\theta$, ϕ and $\phi + d\phi$ (§ 11·2) we shall have

$$d\Omega = \sin\theta \, d\theta \, d\phi \quad . \quad . \quad . \quad (11\cdot901)$$

and we may express $(11\cdot9)$ in the form

$$n' = \frac{n \sin\theta \, d\theta \, d\phi}{4\pi} \quad . \quad . \quad . \quad (11\cdot902)$$

The number of the molecules, travelling in directions included within $d\Omega$, which strike a small element $\mathbf{dS}$ of the wall of the containing vessel in the time dt, will be the same as the number of them contained in the cylinder CBED (Fig. 11·91) the volume of which is

$$\mathbf{c} \, dt \, \mathbf{dS} \cos\theta,$$

and therefore equal to

$$n' \, \mathbf{c} \, dt \, \mathbf{dS} \cos\theta \, . \quad (11\cdot903)$$

Fig. 11·91

Each molecule has the momentum $m\mathbf{c}$ with a normal component $m\mathbf{c}\cos\theta$, and on collision with $\mathbf{dS}$ this will be reversed, so that if we confine our attention to this component—and we may do so since it is the force normal to $\mathbf{dS}$ that we are investigating —the change of momentum which a single molecule suffers on collision will be

$$2m\mathbf{c}\cos\theta,$$
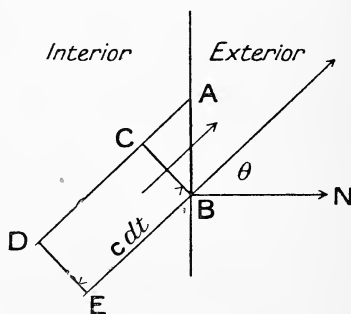
and consequently in the time dt the total change of momentum due to collisions with $\mathbf{dS}$ will be

$$2mn'\mathbf{c}^2\,\mathbf{dS}\,dt\cos^2\theta \quad . \quad . \quad . \quad (11\cdot904)$$

Substituting in this the expression $(11\cdot902)$ for n', we have

$$\frac{mn\mathbf{c}^2}{2\pi}\,\mathbf{dS}\,dt\cos^2\theta\sin\theta\,d\theta\,d\phi.$$

The change of momentum per unit time, or the contribution of these collisions to the force on $\mathbf{dS}$, is therefore

$$\frac{mn\mathbf{c}^2}{2\pi}\,\mathbf{dS}\cos^2\theta\sin\theta\,d\theta\,d\phi,$$

and we obtain for the force due to all the molecules with the velocity $\mathbf{c}$,

$$\frac{mn\mathbf{c}^2}{2\pi}\mathbf{dS}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{\phi=0}^{\phi=2\pi}\cos^2\theta\sin\theta\,d\theta\,d\phi,$$

or

$$\frac{mn\mathbf{c}^2}{3}\mathbf{dS}.$$

The kinetic energy of a single molecule is $\dfrac{m\mathbf{c}^2}{2}$, and that of the n molecules in the unit volume, $\dfrac{mn\mathbf{c}^2}{2}$; so that their contribution to the force on $\mathbf{dS}$ is

$$\tfrac{2}{3}K\mathbf{dS},$$

where K is the kinetic energy per unit volume.

It is now evident that, whatever may be the law of distribution of velocities, the total force on $\mathbf{dS}$ will be given by

$$\tfrac{2}{3}(K_1 + K_2 + \ldots)\mathbf{dS},$$

where K_1, K_2, K_3, etc., are the kinetic energies per unit volume of the molecules with the velocities $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, etc., respectively. We have consequently for the pressure

$$p = \tfrac{2}{3}\ (kinetic\ energy\ per\ unit\ volume), \quad . \quad (11\cdot91)$$

or $\qquad\qquad p v = \tfrac{2}{3}$ (total kinetic energy in the gas) $\quad . \quad (11\cdot911)$

The factitious assumption about the nature of the wall of the containing vessel is not necessary. If we consider, in the first place, only the molecules travelling up to the wall, and work out by the method just described the sum of the components of their momenta perpendicular to $\mathbf{dS}$, we shall find for their contribution to the pressure—whatever happens to them on colliding with the wall—

$$p = \tfrac{1}{3}\ (kinetic\ energy\ per\ unit\ volume),$$

and the assumption that the directions of the velocities, **c**, are uniformly distributed leads to the same contribution from the momenta leaving the wall. The two contributions combined yield the amount expressed by (**11·91**). The identification of heat and kinetic energy will make the right-hand side of (**11·911**) constant so long as the temperature is constant, and we thus have an explanation of **Boyle's law**.

A very little reflexion will show that (**11·91**) is true for a mixture of gases. Therefore in such a case

$$p = \tfrac{2}{3}(L_1 + L_2 + L_3 + \ldots),$$

where L_1, L_2, L_3 . . . represent the kinetic energy per unit volume of the constituent gases of the mixture and consequently

$$p = p_1 + p_2 + p_3 + \ldots \ldots \quad (11·912)$$

In this formula

$$p_1 = \tfrac{2}{3}L_1,$$
$$p_2 = \tfrac{2}{3}L_2,$$
$$p_3 = \tfrac{2}{3}L_3, \quad \ldots \ldots \quad (11·913)$$

etc., represent the contributions of the constituent gases 1, 2, 3 . . . respectively to the total pressure. In other words, p_1, p_2, p_3 . . . are the **partial pressures** of the constituent gases, and (**11·912**) asserts that the total pressure is equal to the sum of the partial pressures. This is equivalent to the statement that the total pressure is equal to the sum of the pressures which each constituent gas would exert if it alone were occupying the volume of the mixture. This is **Dalton's law of partial pressures.**

Since (**11·91**) is equivalent to

$$p = \frac{\varrho \overline{c^2}}{3},$$

where ϱ is the density of the gas, we can easily find $\sqrt{\overline{c^2}}$, or the root of the mean square of the velocity of translation of the molecules of any gas. For hydrogen, oxygen and nitrogen at normal temperature and pressure we find $1·844 \times 10^5$, $4·61 \times 10^4$ and $4·92 \times 10^4$ cm. sec.$^{-1}$ respectively.

§ 12. Laws of Charles and Avogadro—Equipartition of Energy

It will have been noted that the kinetic energy referred to in (**11·911**) is that of translation only. The molecules may however have kinetic energy of rotation as well. To begin with let us imagine them to be perfectly smooth spheres. Any force exerted on such a molecule must be normal to its surface, and

consequently passes through its centre and therefore, if it is uniform, through its centre of mass. It follows that its kinetic energy of rotation cannot undergo any change. A gas constituted of such molecules will behave in precisely the same way whether they have rotational kinetic energy or not. It can have no observational consequences. Each of such molecules may be regarded as a dynamical system with three q's and the corresponding p's, the former being the co-ordinates of the centre of the molecule and the latter the associated momenta. Such molecules have *virtually* only three **degrees of freedom.** In the next place let us suppose the molecules to be perfectly smooth and uniform ellipsoids of revolution. The forces experienced by them in collisions must again be normal to the ellipsoidal surfaces. They will therefore always pass through the axis of revolution, but *not necessarily* through the centre or centre of mass of the ellipsoid. What has been said about the rotational kinetic energy of the spherical molecules applies to the kinetic energy of rotation about the axes of revolution of the ellipsoidal molecules, but not to the rotational kinetic energy about other axes. The ellipsoidal molecule consequently has virtually five q's or degrees of freedom ; three to fix the position of its centre of mass, and two to fix the direction of its axis of revolution. A sixth q representing angular displacements about this axis is not associated with any observable consequences and is for us virtually non-existent.

Quite obviously the kinetic energy of translation will vary greatly from one molecule to another and the average kinetic energy of translation, reckoned for a volume so minute that it contains only one or two molecules, will likewise differ very much in different parts of the gas, and at different times. For the present we shall make the following assumption, leaving the discussion of its validity till a later stage. When the temperature of the wall of the containing vessel is kept constant the gas reaches in time a final state which we shall describe as one of **statistical equilibrium,** and when this has been attained the average kinetic energy of translation calculated for the molecules in any volume at a given instant approaches a limiting value as the volume taken is made sufficiently big. We assume further that this limiting value is already practically reached while the volume in question is still so small a fraction of the space occupied by the gas that it may be treated as an element $dx\,dy\,dz$. For brevity we may say that statistical equilibrium is associated with a uniform distribution of translational kinetic energy among the molecules. From another point of view, statistical equilibrium is associated with uniform temperature throughout the

gas and we shall define a scale of temperature by the statement

$$\bar{K} = aT \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (12)$$

where $\bar{K}$ means the average translational kinetic energy of a molecule, and a is a constant, the same for all gases.[1] This definition leaves the unit of temperature difference still undefined. It follows from (11·911) that

$$pv = \tfrac{2}{3}NaT,$$

or, if we write k for $\tfrac{2}{3}a$,

$$pv = NkT \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (12 \cdot 01)$$

If two different gases occupy equal volumes at the same pressure and temperature it follows, since k is a universal constant, that both contain the same number of molecules. This is **Avogadro's law**. Furthermore if pv has the same value for a number of different gases all at the same temperature, e.g. that of melting ice, it will necessarily have the same value for all the gases at any other temperature, e.g. that of saturated steam at normal pressure. This is the **law of Charles**. Equation (12·01) really unites in one statement the laws of Boyle, Charles and Avogadro. The unit of temperature difference is usually fixed by making the difference between the temperature of saturated steam under normal pressure, and that of melting ice, 100. If therefore we write (12·01) in the form

$$pv = RT \qquad . \qquad . \qquad . \qquad . \qquad (12 \cdot 011)$$

the unit of temperature difference is fixed by

$$R = \frac{(pv)_1 - (pv)_0}{100} \qquad . \qquad . \qquad . \qquad (12 \cdot 012)$$

where $(pv)_1$ and $(pv)_0$ mean the values of pv at the temperatures named above respectively.

The definition of temperature adopted above is justified by its consequences; but it involves an assumption which it is desirable we should be able to deduce as a consequence of the statistical equilibrium of a large number of dynamical systems, namely that the average kinetic energy of translation of all molecules, however much these may differ from one another, is the same when statistical equilibrium has been set up. We shall later establish a theorem which contains this as a special case, namely that the average kinetic energy per degree of freedom is the same for all molecules in statistical equilibrium. This is called the theorem of **equipartition of energy**. Since the average kinetic energy of translation of a molecule is $\tfrac{3}{2}kT$, and this is distributed over three degrees of freedom, it follows

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[1] This assumption will be justified later.

that the average kinetic energy associated with any one q or degree of freedom is $\frac{1}{2}kT$ per molecule.

The average kinetic energy of a molecule with ν degrees of freedom is consequently $\frac{\nu}{2}kT$. If we assume the potential energy of the molecules of a gas to be an invariable quantity, we find for the energy of a gram of the gas

$$E = \frac{\nu}{2}NkT + \text{constant} \quad . \quad . \quad . \quad . \quad (12 \cdot 02)$$

where N is the number of molecules in a gram. The specific heat of the gas at constant volume is therefore

$$c_v = \frac{dE}{dT} = \frac{\nu}{2}Nk,$$

or
$$c_v = \frac{\nu}{2}R \quad . \quad . \quad . \quad . \quad . \quad (12 \cdot 021)$$

where R is the gas constant for a gram. If the volume of the gas is changed, a certain amount of work, positive or negative according as the volume increases or diminishes, will be done by the pressure. The force exerted on an element $\mathbf{dS}$ of the wall of the vessel will be $p\mathbf{dS}$, and if $\mathbf{dS}$ is displaced a distance $\mathbf{dl}$ the work done will be

$$p(\mathbf{dS} \; \mathbf{dl}) \; ;$$

and consequently during a small expansion of the vessel the work done will be

$$p\Sigma(\mathbf{dS} \; \mathbf{dl}),$$

the summation being extended over all the elements $\mathbf{dS}$ which make up the surface of the containing wall. The work done may therefore be expressed in the form

$$pdv.$$

This gives us, for any change in volume from an initial value v_1 to a final value v_2, the expression

$$W = \int_{v_1}^{v_2} pdv \quad . \quad . \quad . \quad (12 \cdot 022)$$

which becomes

$$p(v_2 - v_1),$$

if the pressure be kept constant during the expansion, or

$$R(T_2 - T_1),$$

by $(12 \cdot 011)$. Consequently the work done is equal to R, if the

associated rise in temperature amounts to one degree. A word of warning is needed here. Equation (12·011) was deduced on the assumption that the volume occupied by the gas was not varying. Consequently the last result will only be valid if the expansion is taking place very slowly, so that the pressure, p, is not sensibly different from the pressure that would exist if the volume were not changing at all. Such an expansion is called a **reversible** expansion. Reversible processes will be discussed in detail in the chapters devoted to thermodynamics. We see now that the specific heat of the gas at constant pressure will exceed that at constant volume by the amount R. Therefore

$$c_p = \frac{\nu}{2}R + R \quad . \quad . \quad . \quad . \quad (12\cdot023)$$

and consequently the ratio

$$\gamma = \frac{c_p}{c_v} = 1 + \frac{2}{\nu} \quad . \quad . \quad . \quad . \quad (12\cdot03)$$

This formula is in good accord with the experimental values for gases the chemical properties of which indicate a relatively simple molecular constitution. The smallest possible number of degrees of freedom is three giving $\gamma = 1\frac{2}{3}$, a value found experimentally for mercury vapour, helium and argon. The value $\gamma = 1\frac{2}{5}$ is found for gases like hydrogen, oxygen and nitrogen, the chemical behaviour of which shows them to consist of molecules having two atoms, while the ratio γ is found to approach more closely still to unity with increasing complexity of molecular structure.

§ 12·1. Maxwell's Law of Distribution

We shall now inquire about the distribution of velocities among the molecules of a gas of the type described in § 11·9. Representing the velocity of an individual molecule by $\mathbf{c} \equiv (u, v, w)$ we have to try to answer the question: **Among the N molecules constituting a gas, how many have velocities lying between the limits $\mathbf{c} \equiv (u, v, w)$ and $\mathbf{c} + d\mathbf{c} \equiv (u + du, v + dv, w + dw)$?** All the molecules with the same absolute velocity $\mathbf{c}$ will be assumed to be uniformly distributed as regards direction (§ 11·9). Let us represent the velocities of the individual molecules by points on a diagram (Fig. 12·1), the X, Y and Z coordinates of any one of these points being numerically equal to the u, v and w respectively of the molecule which the point represents. Imagine two infinite and parallel planes perpendicular to the X axis and cutting it at u and $u + du$. The

number of representative points between the two planes may be expressed in the form

$$Nf(u)du,$$

$f(u)$ being an unknown function of u. If we construct two further parallel planes, v and $v + dv$, similarly related to the Y axis, and if we make the very reasonable assumption that the number of molecules with Y components of velocity lying between v and $v + dv$ is quite independent of their X components of velocity, we may write for the number of molecules with representative points in the region □, bounded by the four planes

$$N'f(v)dv,$$

where N' is the number of representative points between the
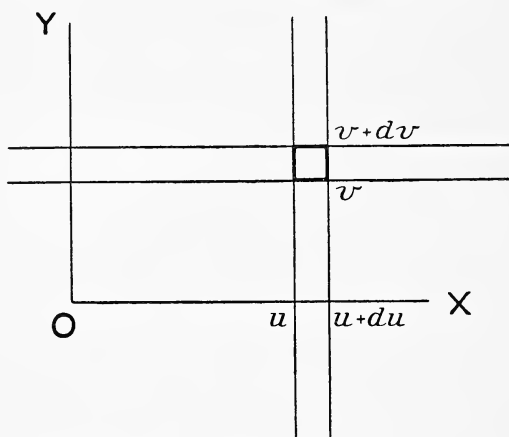
Fig. 12·1

planes u and $u + du$. Therefore the number of representative points in this region is

$$Nf(u)f(v)du\ dv.$$

Finally we may imagine a third pair of planes perpendicular to the Z axis at w and $w + dw$, and we find for the number of representative points in the small volume $du\ dv\ dw$ enclosed by the three pairs of planes

$$Nf(u)f(v)f(w)du\ dv\ dw \quad . \quad . \quad . \quad (12\cdot1)$$

Our problem is to find out, if possible, the character of the function f. The product $Nf(u)f(v)f(w)$ in $(12\cdot1)$ is the number of points per unit volume at (u, v, w) in the space of Fig. 12·1 which represent molecules. At all points on the surface of a

sphere of radius **c**, and having its centre at the origin, this product must have the same value, i.e.

$$f(u)f(v)f(w) = \text{const.},$$

and, of course,

$$u^2 + v^2 + w^2 = c^2 = \text{const.}$$

If therefore (u, v, w) and $(u + du, v + dv, w + dw)$ are neighbouring points on the spherical surface,

$$f'(u)f(v)f(w)du + f(u)f'(v)f(w)dv + f(u)f(v)f'(w)dw = 0,$$

and

$$udu + vdv + wdw = 0.$$

In the former of these equations $f'(u)$ is an abbreviation for $\dfrac{df(u)}{du}$. We may write these equations in the form

$$\frac{f'(u)}{f(u)}du + \frac{f'(v)}{f(v)}dv + \frac{f'(w)}{f(w)}dw = 0$$

$$udu + vdv + wdw = 0 \quad . \quad . \quad (12\cdot11)$$

Multiply the second equation by λ and subtract. The result will be

$$\left(\frac{f'(u)}{f(u)} - \lambda u\right)du + \left(\frac{f'(v)}{f(v)} - \lambda v\right)dv + \left(\frac{f'(w)}{f(w)} - \lambda w\right)dw = 0,$$

and if we choose λ so that

$$\frac{f'(u)}{f(u)} - \lambda u = 0,$$

we shall be left with

$$\left(\frac{f'(v)}{f(v)} - \lambda v\right)dv + \left(\frac{f'(w)}{f(w)} - \lambda w\right)dw = 0.$$

In this last equation it is evident that dv and dw are arbitrary and their coefficients must consequently be zero. In this way we get the three equations

$$\frac{f'(u)}{f(u)} = \lambda u,$$

$$\frac{f'(v)}{f(v)} = \lambda v,$$

$$\frac{f'(w)}{f(w)} = \lambda w. \quad . \quad . \quad . \quad . \quad (12\cdot12)$$

The factor λ must be a constant because the first equation represents it as a function of u only, the second one as a function

of v only and so on.   The equations $(12\cdot12)$ are equivalent to

$$\frac{d}{du}\{\log f(u)\} = \lambda u,$$

$$\frac{d}{dv}\{\log f(v)\} = \lambda v,$$

$$\frac{d}{dw}\{\log f(w)\} = \lambda w.$$

We thus get

$$\log f(u) = \frac{\lambda}{2}u^2 + \text{const.},$$

or, if we replace $\frac{\lambda}{2}$ by $-a$,

$$f(u) = Ae^{-au^2} \quad . \quad . \quad . \quad . \quad . \quad (12\cdot13)$$

The following definite integrals find frequent application in the kinetic theory.   If ε be a positive constant and n a positive integer,

$$J_1 = \int_0^\infty e^{-\varepsilon x}dx = \varepsilon^{-1},$$

$$J_2 = \int_0^\infty e^{-\varepsilon x^2}dx = \tfrac{1}{2}\pi^{\frac{1}{2}}\varepsilon^{-\frac{1}{2}},$$

$$J_{1n} = \int_0^\infty x^n e^{-\varepsilon x}dx = n!\varepsilon^{-(n+1)},$$

$$J_{2n} = \int_0^\infty x^{2n}e^{-\varepsilon x^2}dx = \frac{1.3.5\ldots(2n-1)}{2^{n+1}}\pi^{\frac{1}{2}}\varepsilon^{-\left(\frac{2n+1}{2}\right)} \quad (12\cdot131)$$

The last two, J_{1n} and J_{2n}, are derived by successive differentiation from J_1 and J_2 respectively.

The constant, A, in $(12\cdot13)$ can be expressed in terms of the constant a.   The expression

$$N\int_{-\infty}^{+\infty} f(u)du$$

must be equal to the number of molecules in the gas, i.e. equal to N.   Therefore

$$A\int_{-\infty}^{+\infty} e^{-au^2}du = 1,$$

and by the second equation (12·131)
$$A \pi^{\frac{3}{2}} \alpha^{-\frac{3}{2}} = 1$$

or
$$A = \alpha^{\frac{3}{2}} \pi^{-\frac{3}{2}} \quad . \quad . \quad . \quad (12 \cdot 14)$$

If we use rectangular co-ordinates, the number of molecules in the element of volume $du\,dv\,dw$ of the representative space of Fig. 12·1 is
$$NA^3 e^{-a(u^2+v^2+w^2)} du\,dv\,dw \quad . \quad . \quad (12 \cdot 15)$$

or
$$NA^3 e^{-a\mathbf{c}^2} \mathbf{c}^2 d\mathbf{c} \sin\theta\,d\theta\,d\phi \quad . \quad . \quad (12 \cdot 151)$$

if we use polar co-ordinates, $\mathbf{c}$, θ and ϕ. The average kinetic energy of translation, $\bar{K}$, of a molecule is evidently given by

$$N\bar{K} = NA^3 \int_0^\infty \int_0^\pi \int_0^{2\pi} \tfrac{1}{2} m\mathbf{c}^2 e^{-a\mathbf{c}^2} \mathbf{c}^2 d\mathbf{c} \sin\theta\,d\theta\,d\phi,$$

or
$$\bar{K} = 2\pi m A^3 \int_0^\infty \mathbf{c}^4 e^{-a\mathbf{c}^2} d\mathbf{c},$$

and by (12·131)
$$\bar{K} = 2\pi m A^3 \frac{1 \cdot 3}{2^3} \pi^{\frac{1}{2}} \alpha^{-\frac{5}{2}},$$

or
$$\bar{K} = 2\pi m \alpha^{\frac{3}{2}} \pi^{-\frac{3}{2}} \cdot \tfrac{3}{8} \pi^{\frac{1}{2}} \alpha^{-\frac{5}{2}},$$

and therefore
$$\bar{K} = \tfrac{3}{4} m \alpha^{-1} \quad . \quad . \quad . \quad (12 \cdot 152)$$

Another way of expressing this is to say that
$$\alpha = \frac{\tfrac{3}{4} m}{\text{Average kinetic energy per molecule}} \quad (12 \cdot 153)$$

If we take the average kinetic energy of translation of a molecule to be $\tfrac{3}{2} kT$ (see § 12 ; the constant α of equation (12) must not be confused with the constant α of equation 12·152) we get from (12·152)
$$\tfrac{3}{2} kT = \tfrac{3}{4} m \alpha^{-1}$$

and therefore
$$\alpha = \frac{m}{2kT} \quad . \quad . \quad . \quad . \quad (12 \cdot 154)$$

We thus find for the number of molecules per unit volume of the representative space

$$N \sqrt{\frac{m^3}{8\pi^3 k^3 T^3}} \, e^{-\frac{m\mathbf{c}^2}{2kT}} \quad . \quad . \quad . \quad (12 \cdot 16)$$

This is **Maxwell's law of distribution of velocities.**

Starting from (12·151), we can easily find $\bar{c}$ the average

of the absolute values of the molecular velocities. This is obviously given by

$$\bar{c} = A^3 \int_0^\infty \int_0^\pi \int_0^{2\pi} c e^{-ac^2} c^2 dc \sin\theta \, d\theta \, d\phi$$

or

$$\bar{c} = 4\pi A^3 \int_0^\infty c^3 e^{-ac^2} dc$$

$$= \frac{4\pi A^3}{2} \int_0^\infty c^2 e^{-ac^2} d(c^2)$$

$$= 2\pi A^3 \int_0^\infty x e^{-ax} dx.$$

By (12·131) this is

$$\bar{c} = 2\pi A^3 a^{-2}$$

and since $A = a^{\frac{1}{2}}\pi^{-\frac{1}{2}}$

$$\bar{c} = 2\pi a^{\frac{3}{2}}\pi^{-\frac{3}{2}}a^{-2}$$

or

$$\bar{c} = 2\pi^{-\frac{1}{2}}a^{-\frac{1}{2}}$$

Therefore

$$(\bar{c})^2 = \frac{4}{a\pi} \qquad \ldots \qquad (12 \cdot 17)$$

But we have seen that

$$\bar{K} = \tfrac{3}{4}\frac{m}{a},$$

or

$$\tfrac{1}{2}m\overline{c^2} = \tfrac{3}{4}\frac{m}{a},$$

and therefore

$$\overline{c^2} = \frac{3}{2a} \qquad \ldots \qquad (12 \cdot 171)$$

so that (12·17 and 12·171)

$$\overline{c^2} = \frac{3\pi}{8}(\bar{c})^2 \qquad \ldots \qquad (12 \cdot 172)$$

That is to say the average of the squares of the velocities is equal to the product of $\dfrac{3\pi}{8}$ and the square of the average velocity.

The averages just calculated are those of quantities associated with the molecules occupying some definite volume at a given instant of time. There are certain other averages of interest

and importance, for instance the average kinetic energy of translation of the molecules passing per unit time (or during a given time) through an element of area $\mathbf{dS}$ from one side to the other. This will obviously be greater than $3kT/2$ because the energies of the faster moving molecules will appear more frequently in the sum from which the average is computed. The number of molecules per unit volume, the velocities of which lie between the limits $\mathbf{c}$ and $\mathbf{c} + \mathbf{dc}$, in absolute value and between θ, ϕ and $\theta + d\theta$, $\phi + d\phi$ in direction (Fig. 11·91) is

$$n' = nA^3 e^{-a\mathbf{c}^2}\mathbf{c}^2 d\mathbf{c} \sin\theta \, d\theta \, d\phi,$$

n being the total number of molecules per unit volume. The number passing through $\mathbf{dS}$ (Fig. 11·91) in the time dt is (11·903)

$$n'\mathbf{c} \, dt \, \mathbf{dS} \cos\theta$$

or $nA^3\mathbf{c}^3 e^{-a\mathbf{c}^2}d\mathbf{c} \sin\theta \cos\theta \, d\theta . d\phi . \mathbf{dS} \, dt.$

The translational kinetic energy transported by them is got by multiplying this expression by $\frac{1}{2}m\mathbf{c}^2$. Therefore the number passing through the unit area per second is

$$\pi nA^3 \int_0^\infty \mathbf{c}^3 e^{-a\mathbf{c}^2}d\mathbf{c}, \quad . \quad . \quad . \quad . \quad (12\cdot18)$$

the integration with respect to the variables θ and ϕ extending from 0 to $\pi/2$ and 0 to 2π respectively. Writing $\mathbf{c}^2 = x$, this becomes

$$\frac{\pi nA^3}{2} \int_0^\infty xe^{-ax}dx \quad . \quad . \quad . \quad (12\cdot181)$$

For the kinetic energy passing through the unit area per second we get in a similar way

$$\frac{\pi mnA^3}{4} \int_0^\infty x^2 e^{-ax}dx \quad . \quad . \quad . \quad (12\cdot182)$$

On evaluating the integral in (12·181) we get for the number of molecules passing through the unit area from one side to the other per unit time

$$\frac{\pi n}{2}a^{\frac{3}{2}}\pi^{-\frac{3}{2}}a^{-2},$$

since $A = a^{\frac{1}{2}}\pi^{-\frac{1}{2}}$; and on substituting $\dfrac{m}{2kT}$ for a we finally obtain

$$n\sqrt{\frac{kT}{2\pi m}} \quad . \quad . \quad . \quad . \quad (12\cdot19)$$

The average kinetic energy of translation of these molecules is given by dividing (12·182) by (12·181). This yields

$$\bar{K}' = ma^{-1} \quad . \quad . \quad . \quad . \quad (12 \cdot 191)$$

On comparing this with $\bar{K}$ (12·152), we see that

$$\bar{K}' = \frac{4}{3}\bar{K} \quad . \quad . \quad . \quad . \quad . \quad (12 \cdot 192)$$

§ 12·2. MOLECULAR COLLISIONS—MEAN FREE PATH

Let us assume the molecules to be spheres, each having a diameter, σ, very small compared with the average distance travelled by any molecule between consecutive collisions in which it is involved. This average distance is called the **mean free path** and we shall define it precisely as the quotient of the sum of the lengths of all the free paths completed during a given interval of time and the number of these paths. The given interval of time is understood to be so long that the quotient of total distance and number of paths is independent of its duration. There are of course several alternative definitions. If we take a given *instant* of time and consider the distance traversed by a molecule between this instant and the instant of its next collision, the average of these distances for all the molecules is the mean free path as defined by Tait. The former definition gives us, as we shall see,

$$\lambda = \frac{1}{\sqrt{2} \mid \pi\sigma^2 n},$$

whereas Tait's definition leads to

$$\lambda_T = \frac{\cdot 677 \ldots}{\pi\sigma^2 n}.$$

In each case n is the number of molecules per unit volume.

We shall begin the attack on the problem of calculating the mean free path by considering the mean of the free paths, described in a given time, by a molecule moving with a speed which is very high compared with that of the vast majority of the remaining molecules. In this calculation we may suppose the remaining molecules to be at rest. Let the velocity of the moving molecule be **c** and consider a cylinder the axis of which is the path of the centre of the moving molecule, and the section of which is a circle of radius σ. Collisions will occur between the moving molecule and all those which have their centres within this cylinder. The length of the cylinder described per unit time will be **c** and its volume $\pi\sigma^2 c$. Hence the number of collisions per unit time will be $\pi\sigma^2 n\mathbf{c}$. Dividing the total distance,

c, which the molecule has travelled, by the number of collisions, we get

$$\lambda_0 = \frac{1}{\pi\sigma^2 n} \qquad \cdots \cdots \quad (12\cdot2)$$

This will at all events give the order of magnitude of the mean free path. We can easily see that the exact expression for the mean free path, calculated in accordance with the definition we have laid down, must represent a number between that just given (12·2) and zero; since a very slowly moving molecule must, so long as it is moving slowly, describe very short free paths.

Let r be the velocity of a molecule, B, *relative to another molecule*, A, the absolute velocities of A and B being c_1 and c_2 respectively. If these two velocities be represented diagrammatically by lines of length c_1 and c_2, as in Fig. 12·2, it will be obvious that
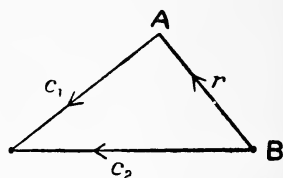
$$r = \{c_2^2 + c_1^2 - 2c_2 c_1 \cos\theta\}^{\frac{1}{2}},$$

since the distance between B and A will be shortened by this amount during one second. The average value of r for a single molecule B with a velocity c_2 and a large number of molecules, A, each with the velocity c_1, and uniformly distributed in direction, will be

Fig. 12·2

$$\bar{r} = \frac{1}{2} \int_0^\pi \sin\theta \, d\theta \{c_2^2 + c_1^2 - 2c_2 c_1 \cos\theta\}^{\frac{1}{2}}.$$

The successive steps in the evaluation of this integral may be written down without detailed explanation as follows :

$$\bar{r} = \int_0^\pi 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} d\left(\frac{\theta}{2}\right)\left\{(c_2 - c_1)^2 + 4c_2 c_1 \sin^2\frac{\theta}{2}\right\}^{\frac{1}{2}},$$

$$\bar{r} = \int_0^1 2x \, dx \{(c_2 - c_1)^2 + 4c_2 c_1 x^2\}^{\frac{1}{2}},$$

$$\bar{r} = \int_0^1 dy \{(c_2 - c_1)^2 + 4c_2 c_1 y\}^{\frac{1}{2}},$$

and therefore

$$\bar{r} = \frac{3c_2^2 + c_1^2}{3c_2}, \quad c_2 > c_1,$$

$$\bar{r} = \frac{3c_1^2 + c_2^2}{3c_1}, \quad c_2 < c_1 \qquad \cdots \quad (12\cdot21)$$

Clausius obtained an approximate expression for the mean free path by assuming all the molecules to have the same velocity, $\bar{c}$, and to be uniformly distributed in direction. With this assumption (**12·21**) becomes

$$\bar{r} = \frac{4}{3}\bar{c} \quad . \quad . \quad . \quad . \quad . \quad (12\cdot211)$$

which represents the average velocity of any one molecule relative to all the others. To get the number of collisions experienced by a particular molecule during the unit time we may suppose it to be moving with the velocity $\bar{r}$ and all the other molecules to be at rest. The method by which (**12·2**) was reached now gives us for the number of collisions per unit time

$$\pi\sigma^2 n\bar{r},$$

while the actual distance travelled by the moving molecule is $\bar{c}$. We get therefore for this approximation to the mean free path

$$\lambda_c = \frac{\bar{c}}{\pi\sigma^2 n\bar{r}},$$

or

$$\lambda_c = \frac{3}{4\pi\sigma^2 n} \quad . \quad . \quad . \quad . \quad . \quad (12\cdot22)$$

by (**12·211**).

In arriving at (**12·2**) and (**12·22**) it has been tacitly assumed that the cylinder of volume $\pi\sigma^2 c$ or $\pi\sigma^2\bar{r}$ is straight. Actually it has a more or less sharp bend at each collision. It is obvious however that this will not cause the expressions $\pi\sigma^2\bar{c}$ or $\pi\sigma^2\bar{r}$ to be in error, since the space swept out will be equal to the sum of the volumes of a large number of cylinders of cross-section $\pi\sigma^2$; the sum of their lengths being $\bar{c}$ or $\bar{r}$ as the case may be.

The way leading to an exact expression for the mean free path, according to the definition we have adopted, is now clearly indicated. From (**12·21**), and Maxwell's law of distribution, we get for the average velocity of a molecule with the absolute velocity c_2, relative to all the other molecules, the expression

$$\bar{r} = 4\pi A^3 \int\limits_0^{c_2} \frac{3c_2{}^2 + c_1{}^2}{3c_2} c_1{}^2 e^{-ac_1{}^2} dc_1$$

$$+ 4\pi A^3 \int\limits_{c_2}^{\infty} \frac{3c_1{}^2 + c_2{}^2}{3c_1} c_1{}^2 e^{-ac_1{}^2} dc_1 \qquad (12\cdot23)$$

Therefore the number of collisions made by it in the unit time will be

$$\pi n\sigma^2\bar{r} \quad . \quad . \quad . \quad . \quad . \quad (12\cdot231)$$

where $\bar{r}$ is given by (12·23). The number of collisions made by all the molecules N will consequently be

$$\nu = \pi n \sigma^2 \int_0^\infty \bar{r} 4\pi A^3 N c_2{}^2 e^{-ac_2{}^2} dc_2$$

or, written out in full,

$$\nu = 16\pi^3 \sigma^2 n N A^6 \left[\int_0^\infty c_2{}^2 e^{-ac_2{}^2} dc_2 \int_0^{c_2} \frac{3c_2{}^2 + c_1{}^2}{3c_2} c_1{}^2 e^{-ac_1{}^2} dc_1 \right.$$

$$\left. + \int_0^\infty c_2{}^2 e^{-ac_2{}^2} dc_2 \int_{c_2}^\infty \frac{3c_1{}^2 + c_2{}^2}{3c_1} c_1{}^2 e^{-ac_1{}^2} dc_1 \right],$$

or

$$\nu = 16\pi^3 \sigma^2 n N A^6 \{C + D\} \quad . \quad . \quad (12\cdot232)$$

In the integrals C and D the integration with respect to c_1 has to be carried out first. If in C the integration with respect to c_2 were carried out first, it would have to be written in the form

$$C = \int_0^\infty c_1{}^2 e^{-ac_1{}^2} dc_1 \int_{c_1}^\infty \frac{3c_2{}^2 + c_1{}^2}{3c_2} c_2{}^2 e^{-ac_2{}^2} dc_2$$

since $c_2 \geq c_1$. It is obvious now that $C = D$ and (12·232) becomes

$$\nu = 32\pi^3 \sigma^2 n N A^6 D \quad . \quad . \quad . \quad (12\cdot233)$$

On evaluation D becomes

$$\pi^{\frac{1}{2}} 2^{-\frac{7}{2}} a^{-\frac{7}{2}},$$

and for the number of collisions which N molecules experience in the unit time, we find

$$\nu = 32\pi^3 \sigma^2 n N A^6 \pi^{\frac{1}{2}} 2^{-\frac{7}{2}} a^{-\frac{7}{2}},$$

or since $A = a^{\frac{1}{2}} \pi^{-\frac{1}{2}}$,

$$\nu = 2\sqrt{2} \, | \, \pi^{\frac{1}{2}} \sigma^2 n N a^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (12\cdot234)$$

The total length of the paths is

$$N\bar{c} = N 2\pi^{-\frac{1}{2}} a^{-\frac{1}{2}} \quad . \quad . \quad . \quad (12\cdot235)$$

by (12·17), and therefore on dividing (12·235) by (12·234) we find for the mean free path

$$\lambda = \frac{1}{\sqrt{2} \, | \, \pi \sigma^2 n} \quad . \quad . \quad . \quad . \quad (12\cdot24)$$

The following table gives the mean free path in centimetres for a number of gases at normal temperature and pressure.

Gas	$\lambda \times 10^5$
Hydrogen	1·78
Oxygen	1·02
Nitrogen	0·95
Carbon dioxide	0·65

§ 12·3. VISCOSITY—THERMAL CONDUCTIVITY

When a gas is in motion as a whole, we have to distinguish between the velocity of its motion, i.e. the **stream velocity, s,** and the velocity of agitation, **c,** of an individual molecule. Let us represent the components of the stream velocity by u', v' and w' and those of the velocity of agitation of a single molecule by u, v and w as before ; so that

$$\mathbf{s} \equiv (u', v', w'),$$
$$\mathbf{c} \equiv (u, v, w) \quad . \quad . \quad . \quad . \quad . \quad (12\text{·}3)$$

Associated with the flow of a gas in a given direction will be a **stream momentum**

$$\Sigma m\mathbf{s} \equiv (\Sigma mu', \Sigma mv', \Sigma mw') \quad . \quad . \quad (12\text{·}301)$$

where m is the mass of a single molecule. When the stream velocity varies from point to point, frictional or viscous forces will be exerted by one part of the gas on another (§ 11·5). These forces find their expression in terms of a tensor t''_{xx}, t''_{yy}, t''_{xy}, etc. Imagine the Z axis to be directed upwards and the Y and Z components of the stream velocity to be zero, so that everywhere

$$\mathbf{s} \equiv (u', 0, 0),$$

and let u' be a function of z only, so that the stream velocity has the same value at all points in the same horizontal plane. The tensor of § 11·5 now simplifies to the single component t''_{xz}, and by (**11·51**)

$$t''_{xz} = n'\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right)$$

or

$$t''_{xz} = \mu\frac{du'}{dz} \quad . \quad . \quad . \quad . \quad . \quad (12\text{·}31)$$

This is the force per unit area exerted in the X direction over any horizontal plane by the more rapidly flowing medium above, on the less rapidly flowing medium below. The kinetic theory

explains this viscous force in the following way : The molecules above the given horizontal plane have a greater stream momentum, mu', than have those below it. Approximately equal numbers will cross the unit area of the plane in both directions in a given time and the lower portion of the medium will therefore gain stream momentum at the expense of the upper portion. The rate of gain of momentum will be a measure of the force exerted on the gas as whole below the given plane and tending to increase its velocity of flow, or conversely it will measure the force hindering the flow of the gas above the plane.

To get an expression for the viscosity, μ, we first find the amount of stream momentum passing upwards through an element of area dS (AB in Fig. 11·91, the normal, N, having the direction of Z). We shall simplify the calculation by assuming that all the molecules have the same absolute velocity of agitation which we take to be the average of the actual velocities of agitation, or $\bar{c}$. We may use the expression (11·903) for the number of molecules passing through dS in the time dt in directions included within the solid angle $d\Omega (= \sin\theta\, d\theta\, d\phi)$. This has to be multiplied by the stream momentum per molecule. Each molecule on passing dS will have travelled, on the average, a distance l (equal to the mean free path

FIG. 12·3

according to one of the possible definitions) since its last collision, and we may take it to have the stream momentum appropriate to the place of its last collision. If dS be in the plane $z = M = $ const., each of these molecules starts from the plane
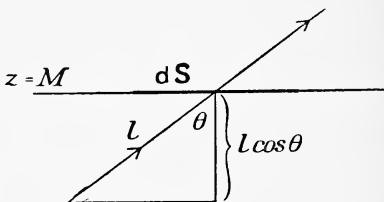
$$z = M - l\cos\theta$$

(Fig. 12·3) and if the stream momentum in the plane $z = M$ be mu', each molecule in question will have the stream momentum

$$m\left\{u' - l\cos\theta\,\frac{du'}{dz}\right\} \quad . \quad . \quad . \quad (12\cdot311)$$

Leaving the first term of this expression on one side for the time being, it becomes

$$- ml\cos\theta\,\frac{du'}{dz}.$$

If we multiply (11·903) by this we get an expression which differs from the corresponding one in § 11·9 only in having

$$- ml\frac{du'}{dz} \text{ replacing } 2mc,$$

so that we have instead of (**11·904**)

$$- ml\frac{du'}{dz}n'\bar{c}\mathbf{d}\mathbf{S}dt\cos^2\theta,$$

or

$$- \frac{mnl\bar{c}}{4\pi}\frac{du'}{dz}\mathbf{dS}\,dt\sin\theta\cos^2\theta\,d\theta\,d\phi \quad. \quad (12·312)$$

Integrating, we get

$$- \frac{mn\bar{c}l}{6}\frac{du'}{dz}\mathbf{dS}\,dt.$$

Therefore the stream momentum carried *upwards* through **dS** in the time dt is

$$\mathbf{P} - \frac{mn\bar{c}l}{6}\frac{du'}{dz}\mathbf{dS}\,dt \quad . \quad . \quad . \quad (12·313)$$

where **P** is the contribution (whatever it may be) due to the term we have left on one side. The corresponding calculation for the stream momentum carried *downwards* obviously gives

$$\mathbf{P} + \frac{mn\bar{c}l}{6}\frac{du'}{dz}\mathbf{dS}\,dt \quad . \quad . \quad . \quad (12·314)$$

Subtracting (**12·313**) from (**12·314**) we get for the net gain of stream momentum by the medium below the plane $z = M$, reckoned per unit area per second,

$$t''_{xz} = \frac{mn\bar{c}l}{3}\frac{du'}{dz}.$$

Consequently

$$\mu = \frac{mn\bar{c}l}{3} \quad . \quad . \quad . \quad . \quad . \quad (12·32)$$

The l in this formula will not be very different from λ_c of (**12·22**), so that we obtain as an approximate expression for the viscosity

$$\mu = \frac{\rho\bar{c}\lambda_c}{3} \quad . \quad . \quad . \quad . \quad (12·321)$$

or, by (**12·22**)

$$\mu = \frac{m\bar{c}}{4\pi\sigma^2} \quad . \quad . \quad . \quad . \quad (12·322)$$

A rigorous calculation of μ for spherical molecules leads to

$$\mu = \cdot350\,\ldots\,\rho\bar{c}\lambda \quad . \quad . \quad . \quad (12·323)$$

where

$$\lambda = \frac{1}{\sqrt{2}\mid\pi\sigma^2 n},$$

and consequently

$$\mu = \frac{\cdot350\,\ldots\,m\bar{c}}{\sqrt{2}\mid\pi\sigma^2} \quad . \quad . \quad . \quad (12·324)$$

It will be seen that the rigorous formulae differ only very little from the approximate expressions. The viscosity, according to (12·322) or (12·324), is equal to $\bar{c}$ multiplied by a constant which depends only on the mass and size of the molecules. The theory indicates therefore that it is proportional to the square root of the absolute temperature and quite *independent of the pressure* of the gas. This relationship was discovered by Clerk Maxwell. Subsequent experiment fully confirmed the independence of the viscosity of the pressure, but it was found to vary more rapidly with the temperature than is indicated by the theory. The discrepancy suggests that the molecular diameter σ depends on the velocity of agitation, that is on the temperature. Sutherland [1] has derived the formula

$$\mu = \mu_0 \left(\frac{T_0 + C}{T + C} \right) \left(\frac{T}{T_0} \right)^{3/2},$$

in which C is a constant characteristic of the gas. This accords well with experimental results.

When the temperature varies from point to point in a gas or any other medium heat flows from places at higher to places at lower temperature. Let us suppose the temperature to have the same value at all points in any plane $z = $ const. There will be a consequent flow of heat in a direction perpendicular to these planes if the temperature varies with z. The **thermal conductivity**, K, is defined by

$$Q = K\frac{dT}{dz} \quad . \quad . \quad . \quad . \quad . \quad (12·33)$$

where Q is the quantity of heat flowing through the unit area per unit time in the direction of *decreasing* values of z. The kinetic theory identifies heat with the kinetic energy of the molecules and the problem of finding an expression for the conductivity of a gas is seen to be mathematically identical with the foregoing problem of viscosity. The kinetic energy per unit mass of the gas is therefore $c_v T$ where c_v is the specific heat of the gas at constant volume, and therefore the average kinetic energy per molecule may be expressed in the form $mc_v T$. This is the quantity transported by a molecule. The identity of the present problem and that of viscosity becomes still more obvious if we represent the temperature by u' instead of T, and the quantity of heat transported through the unit area per unit time by t_{xz} instead of Q. The constant K will then occupy the place of μ in formula (12·31). The quantity

[1] Sutherland : *Phil. Mag.*, **36**, p. 507 (1893).

transported by a single molecule is then $mc_v u'$ instead of mu' as in the viscosity problem. Consequently we find

$$K = \mu c_v \quad . \quad . \quad . \quad . \quad (12\text{·}331)$$

i.e. the thermal conductivity is equal to the product of the viscosity and the specific heat at constant volume. Experiment confirms the proportionality of thermal conductivity and viscosity indicated by (12·331), and that the two quantities vary in the same way with temperature ; but here the agreement ends. It is found in fact that

$$K = a\mu c_v \quad . \quad . \quad . \quad . \quad (12\text{·}332)$$

where a is a constant in the neighbourhood of 2·5 for monatomic gases, such as helium and argon, 1·9 for diatomic gases, such as oxygen, hydrogen and nitrogen, and still smaller for more complex molecules. The discrepancy between the theory given above and experiment is mainly due to the assumption made about the distribution of velocities. A more rigorous theory based on a suitable modification of Maxwell's law of distribution —the existence of the temperature gradient obviously puts the law in error—does in fact yield $a = 2\text{·}5$, 1·9 and 1·75 for molecules with one, two and three atoms respectively.

§ 12·4.   Diffusion of Gases

The phenomenon of the diffusion of one gas into another is analogous to that of the conduction of heat and the definition of the coefficient of diffusion, or the **diffusivity** as it is usually termed, of a gas A into another B is similar to that of thermal conductivity. Instead of a temperature gradient, we now have a concentration gradient ; and instead of considering a transport of heat or kinetic energy we have now to study the passage of the molecules of one gas into the other. Let n_1 and n_2 be the numbers of molecules per unit volume of two gases occupying the same enclosure, and take the case where n_1 and n_2 are functions of one co-ordinate, z, only, just as in the problems of viscosity and conductivity we supposed the stream velocity or the temperature to be functions of z only. If D_{12} represent the diffusivity of gas 1 into gas 2, we have

$$G_1 = - D_{12} \frac{dn_1}{dz},$$

where G_1 is the number of molecules of gas 1 which pass through the unit area perpendicular to the Z axis per second in the direction of increasing z. Similarly

$$G_2 = - D_{21} \frac{dn_2}{dz}.$$

Following the method of calculating the viscosity in § 12·3, we bear in mind that the molecules passing through the element of area dS, in the sense of increasing z for example, and in a direction inclined at an angle θ to the normal (i.e. to Z), have their last collision at the average distance $l \cos \theta$ below dS; so that in evaluating the number passing through dS we must take the concentration appropriate to $z = M - l \cos \theta$ (§ 12·3). For the number of molecules of gas 1 passing through dS in the time dt and travelling in directions included in the solid angle $d\Omega = \sin \theta \, d\theta \, d\phi$, we easily find

$$\bar{c}_1\left(n_1 - l_1 \cos \theta \frac{dn_1}{dz}\right)\frac{d\Omega}{4\pi}dS \, dt \cos \theta,$$

n_1 meaning the concentration in the plane $z = M$, which contains dS. As in the viscosity problem the first term will contribute nothing to the end result, and we are left with

$$-\frac{\bar{c}_1 l_1}{4\pi}\frac{dn_1}{dz} \, dS \, dt \sin \theta \cos^2 \theta \, d\theta \, d\phi,$$

which takes the place of (12·312) in the viscosity problem. On integrating we get

$$-\frac{\bar{c}_1 l_1}{6}\frac{dn_1}{dz} \, dS \, dt,$$

and therefore the number of molecules passing upwards (i.e. in the direction of increasing z) through dS in the time dt will be

$$P - \frac{\bar{c}_1 l_1}{6}\frac{dn_1}{dz} \, dS \, dt.$$

Similarly the number passing downwards will be

$$P + \frac{\bar{c}_1 l_1}{6}\frac{dn_1}{dz} \, dS \, dt.$$

Therefore the net number passing upwards through the unit area per second is

$$-\frac{\bar{c}_1 l_1}{3}\frac{dn_1}{dz} \qquad . \quad . \quad . \quad . \quad . \quad (12\cdot4)$$

The corresponding quantity for the other constituent is

$$-\frac{\bar{c}_2 l_2}{3}\frac{dn_2}{dz} \qquad . \quad . \quad . \quad (12\cdot401)$$

In arriving at these formulae one important circumstance has been neglected. We have tacitly assumed that while gas 1 is diffusing, gas 2 is quiescent. Since however $n_1 + n_2$ remains constant, there must be in general some motion of the gas as a whole. Let us suppose the velocity of this motion (in the

Z direction) to be w'. Then our element dS must be travelling relatively to a *fixed* element dS_0 with the velocity w'. The expressions (**12·4**) and (**12·401**) must therefore be amended as follows :

$$G_1 = -w'n_1 - \frac{\bar{c}_1 l_1}{3}\frac{dn_1}{dz},$$

$$G_2 = -w'n_2 - \frac{\bar{c}_2 l_2}{3}\frac{dn_2}{dz} \quad . \quad . \quad (12·402)$$

Now since $n_1 + n_2$ is constant it follows that

$$G_1 + G_2 = 0$$

and

$$\frac{dn_2}{dz} = -\frac{dn_1}{dz}.$$

Consequently

$$0 = -w'(n_1 + n_2) - \tfrac{1}{3}(\bar{c}_1 l_1 - \bar{c}_2 l_2)\frac{dn_1}{dz}$$

and therefore

$$-w' = \frac{1}{3(n_1 + n_2)}(\bar{c}_1 l_1 - \bar{c}_2 l_2)\frac{dn_1}{dz}.$$

On substituting this expression for $-w'$ in (**12·402**) we get

$$G_1 = -\frac{n_2\bar{c}_1 l_1 + n_1\bar{c}_2 l_2}{3(n_1 + n_2)}\frac{dn_1}{dz}$$

and

$$G_2 = -\frac{n_2\bar{c}_1 l_1 + n_1\bar{c}_2 l_2}{3(n_1 + n_2)}\frac{dn_2}{dz} \quad . \quad . \quad . \quad (12·41)$$

Therefore

$$D_{12} = D_{21} = \frac{n_2\bar{c}_1 l_1 + n_1\bar{c}_2 l_2}{3(n_1 + n_2)} \quad . \quad . \quad . \quad (12·42)$$

If the temperature be kept constant, the ratios $n_2/(n_1 + n_2)$ and $n_1/(n_1 + n_2)$ will remain constant as also $\bar{c}_1$ and $\bar{c}_2$; but l_1 and l_2 will vary inversely as the pressure. So that the diffusivity is inversely proportional to the pressure at constant temperature. When the pressure is kept constant, the ratios $n_2/(n_1 + n_2)$ and $n_1/(n_1 + n_2)$ will again remain constant ; but l_1 and l_2 will be proportional to the temperature while $\bar{c}_1$ and $\bar{c}_2$ are proportional to the square root of the temperature. We conclude therefore that at constant pressure the diffusivity is proportional to $T^{3/2}$. Combining both conclusions we may say that the diffusivity is proportional to

$$\frac{T^{3/2}}{p}.$$

Experiment confirms this result so far as the dependence on the pressure is concerned ; but the diffusivity is found to vary

more rapidly with temperature than the 3/2 law indicates. It will be recollected that a similar relation between experiment and theory was pointed out in connexion with viscosity.

A phenomenon of great interest is the diffusion of a gas through minute apertures in a membrane, or in the wall of the containing vessel. This must be distinguished from the streaming or *effusion* of a gas through apertures which, though small in the ordinary sense, are nevertheless wide enough to permit the simultaneous egress of enormous numbers of molecules. If such an aperture is very short in comparison with its breadth, the velocity in the emerging stream of gas is given approximately by Bernoulli's theorem according to which

$$p_1 - p_2 = \tfrac{1}{2}\rho v_2{}^2 - \tfrac{1}{2}\rho v_1{}^2,$$

p_1 and p_2 and v_1 and v_2 are the pressures and stream velocities at the points 1 and 2 respectively. So that if p_1 is the pressure in the interior of a large vessel where the velocity v_1 is practically zero, and if p_2 is the pressure just outside the aperture, we have

$$p_1 - p_2 = \tfrac{1}{2}\rho v^2,$$

or
$$v = \sqrt{\frac{2(p_1 - p_2)}{\rho}},$$

for the velocity reached in the aperture. This result forms the basis of a simple method of comparing the densities of gases devised by Bunsen. If the aperture is in the nature of a long channel, the streaming through it of the gas is governed approximately by the formula of Poiseuille.

In neither of these cases is there any separation in the case of a mixture of gases. The partial pressures of the gases play no part in the phenomena; but only the total pressure. It is different with true diffusion which depends on the motions of the individual molecules and therefore does not become evident till the openings are so minute that only one or two molecules are passing through them at any one instant. If we have a number of gases (between which we distinguish by subscripts 1, 2, 3, . . . s, . . .) contained in two vessels separated by a partition in which are such minute apertures, and if n_s' and n_s'' represent the numbers of molecules per unit volume of the gas, s, in the two vessels respectively; it is clear that the number of molecules of the gas s, which leave the first vessel per unit time will (other things being equal) be proportional to n_s' and the number leaving the second vessel to n_s''. This is an immediate consequence of (12·19). Other things being equal therefore, the rate of diffusion of a gas (expressed by the number of molecules diffusing in the unit time) is proportional to the difference of its *partial pressures* on the two sides

of the membrane. On the other hand the rates of diffusion of different gases under like conditions are proportional to their mean velocities $\bar{c}_1,\ \bar{c}_2,\ \bar{c}_3,\ \ldots$, and since by § 12

$$m_1\bar{c}_1{}^2 = m_2\bar{c}_2{}^2,$$

and therefore

$$\frac{\bar{c}_1{}^2}{c_2{}^2} = \frac{m_2}{m_1},$$

or by (12·172)

$$\frac{\bar{c}_1}{\bar{c}_2} = \sqrt{\frac{m_2}{m_1}}\Bigg|\ ,$$

it follows that the rates of diffusion are inversely proportional to the square roots of the masses of the molecules and therefore inversely proportional to the square roots of the densities (measured under like conditions of pressure and temperature) of the gases. These deductions are identical with the experimental result known as **Graham's law.**

A membrane or wall which permits only one gas in a mixture to diffuse through it is called **semi-permeable.** For example palladium at a suitable temperature allows hydrogen to diffuse through it ; but not other gases. The picture which the kinetic theory gives us of this state of affairs is that of a partition with apertures so small that the molecules of only one gas are small enough to enter them. Imagine a palladium tube (maintained at a sufficiently high temperature) containing within it, say, nitrogen and surrounded on the outside by hydrogen kept at constant pressure. The latter gas will continue to diffuse inwards until its partial pressure inside is equal to its pressure outside. The excess of the total pressure inside over that outside will therefore be equal to the partial pressure of the non-diffusing gas, or the pressure it would exert if it occupied the palladium vessel alone. Similar phenomena are associated with diffusion in liquids through semi-permeable membranes (made by depositing copper ferrocyanide inside the wall of a vessel of unglazed earthenware). If such a vessel contains an aqueous solution of a crystalline body, sugar for example, and is surrounded by pure water, only the latter diffuses and the excess of the pressure inside the semi-permeable vessel over that outside, when equilibrium has ultimately been reached, is naturally associated with the dissolved sugar and is called its **osmotic pressure.**

§ 12·5. Theory of van der Waals

We have so far supposed the dimensions of the individual molecules to be so small that their total proper volume is a negligible fraction of that of the containing vessel (§ 11·9). Let us now examine some of the consequences which ensue when

this fraction is not negligible. The centres of any two molecules cannot approach nearer to one another than a distance σ equal to the diameter of a molecule. Imagine a sphere of radius σ described round the centre of each molecule in the gas. We shall call such a sphere (after Boltzmann) the *covering sphere* of the molecule. The sum of the volumes of the covering spheres will therefore be $\frac{4}{3}\pi\sigma^3 N$, or $8v$, where N is the total number of molecules and v their total proper volume. Since the centres of a pair of molecules may be separated by as short a distance as σ, some of the covering spheres will overlap ; but this over-lapping volume will be small by comparison with v and we shall neglect it. The part of the total volume V in which it is possible for the centre of a given molecule to be situated may consequently be taken to be

$$V - \frac{4}{3}\pi\sigma^3 N.$$

Let us reconsider, in the light of this result, the deduction of the expression $(11·91)$ for the pressure of the gas. The centres of the molecules on colliding with AB (Fig. 12·5) will reach a plane CD, separated from AB by the distance $\sigma/2$. Let us replace the cylinder BCDE of Fig. 11·91 by the cylinder DCEF of Fig. 12·5, with a perpendicular distance $\mathbf{c}dt \cos \theta$ between its end faces. This cylinder plays exactly the same part in the calculation as the former one, and has the same volume $\mathbf{c}dt d\mathbf{S} \cos \theta$.
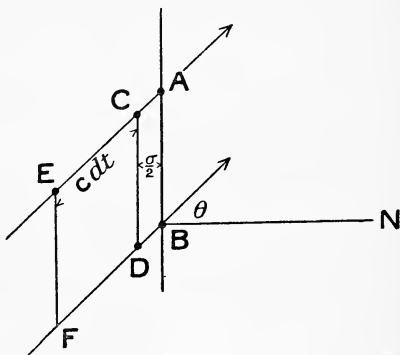
Fig. 12·5

We have to re-calculate n' in $(11·903)$. This formula might have been written

$$\frac{d\Omega}{4\pi} \frac{N}{V} \mathbf{c} \, dt \, d\mathbf{S} \cos \theta \quad . \quad . \quad . \quad . \quad (12·5)$$

where N is the *total number* of molecules, of velocity $\mathbf{c}$. The total number of molecules *per unit volume of the space available for their centres* is now seen to be not $\frac{N}{V}$, but

$$\frac{N}{V - \frac{4}{3}\pi\sigma^3 N} \quad . \quad . \quad . \quad (12·501)$$

If now the cylinder DCEF (Fig. 12·5) were in the *interior of the gas*, the space within it available for the centres of molecules would be

$$\mathbf{c}\, dt\, d\mathrm{S} \cos\theta \; \frac{V - \dfrac{4}{3}\pi\sigma^3 N}{V} \qquad . \quad . \quad (12\text{·}502)$$

If we take dt to be very small indeed, the cylinder will be so narrow that the centres of the molecules, whose covering spheres penetrate the cylinder, will, except for a negligible fraction, lie outside it. We should say that half of these centres were outside EF and the remaining half outside CD (Fig. 12·5). Since however the distance between CD and the wall AB of the vessel is actually only $\dfrac{\sigma}{2}$, no covering spheres of molecules penetrate it from that side at all. The last expression (12·502) must therefore be amended as follows :

$$\mathbf{c}\, dt\, d\mathrm{S} \cos\theta\; \frac{V - \frac{2}{3}\pi\sigma^3 N}{V} \qquad . \quad . \quad (12\text{·}503)$$

To get the number of the N molecules which are in the cylinder DCEF and are moving in the directions included within the limits of the solid angle $d\Omega$ we must therefore multiply together $\dfrac{d\Omega}{4\pi}$ and the expressions (12·501) and (12·503). We thus obtain

$$\frac{d\Omega}{4\pi}\cdot\frac{N}{V}\; \frac{1 - \frac{2}{3}\pi\sigma^3\dfrac{N}{V}}{1 - \dfrac{4}{3}\pi\sigma^3\dfrac{N}{V}}\; \mathbf{c}\, dt\, d\mathrm{S} \cos\theta,$$

which has to take the place of (12·5). The total proper volume of the molecules is

$$v = \frac{\pi\sigma^3}{6}N,$$

so that we get

$$\frac{d\Omega}{4\pi}\frac{N}{V}\; \frac{\left(1 - \dfrac{4v}{V}\right)}{\left(1 - \dfrac{8v}{V}\right)}\; \mathbf{c}\, dt\, d\mathrm{S} \cos\theta,$$

or, neglecting $(v/V)^2$ and higher powers,

$$\frac{d\Omega}{4\pi}\; \frac{N}{V - 4v}\; \mathbf{c}\, dt\, d\mathrm{S} \cos\theta \quad . \quad . \quad . \quad (12\text{·}51)$$

In recalculating the pressure therefore, we learn that the in-

fluence of the size of the molecules is precisely that which might be brought about by a reduction in volume equal to $4v$. If we write, as is usual,

$$b = 4v \quad . \quad . \quad . \quad . \quad . \quad (12 \cdot 52)$$

we must amend (12·011) to read

$$p(V - b) = RT \quad . \quad . \quad . \quad . \quad (12 \cdot 53)$$

A further amendment due, like that just described, to J. D. van der Waals, is based on the supposition that the molecules exert attractive forces on one another which however are only appreciable when the separation of the molecules does not exceed a certain quite small distance R. Any molecule in the interior of the gas will therefore be under the influence of those situated in the sphere of radius R described about this molecule as centre. We may therefore suppose the resultant force exerted on it to be practically zero. It is different in the case of a molecule quite close to the boundary. The attracting molecules are all, or mostly, on one side of it instead of being uniformly distributed in a spherical region round about it. Over the whole boundary of the gas there will be a layer of molecules, extending to a depth R, which experience resultant forces in the direction of the interior of the gas. This will give rise to a pressure over and above that applied through the wall of the vessel or enclosure containing the gas. Since the number of molecules in this layer is practically proportional to the density of the gas, and the same is likewise true of the number of molecules attracting them, it is evident that the additional pressure may be taken as proportional to the square of the density or as inversely proportional to the square of the volume of the gas. We have therefore to amend equation (12·53) by adding to p a term a/V^2, where a is a suitable small constant. In this way we obtain the improved gas equation of van der Waals,

$$\left(p + \frac{a}{V^2} \right) (V - b) = RT \quad . \quad . \quad . \quad (12 \cdot 54)$$

which may also be written in the form

$$V^3 - \left(b + \frac{RT}{p} \right) V^2 + \frac{a}{p} V - \frac{ab}{p} = 0 \quad . \quad (12 \cdot 541)$$

The isothermals (constant temperature curves) according to (12·54) or (12·541) are diagrammatically illustrated in Fig. 12·51. The arrow indicates the order of increasing temperature. The portions of these isothermals which slope downwards from left to right, for example in the isothermal ACEG the portions ABC and EFG, correspond moderately closely with experimentally

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found isothermals (if suitable values are given to a, b and R) the former representing states in which the whole of the substance is in the liquid phase, and the latter those in which the substance is wholly vapour. Those states corresponding to portions of the isothermals, like CDE, which slope upwards from left to right are unstable (which explains why we do not observe them). For consider the state of affairs at such a point as H. A slight increase in the pressure will cause a diminution in volume and, as the slope of the curve indicates, a lower pressure than the original one is now necessary (at
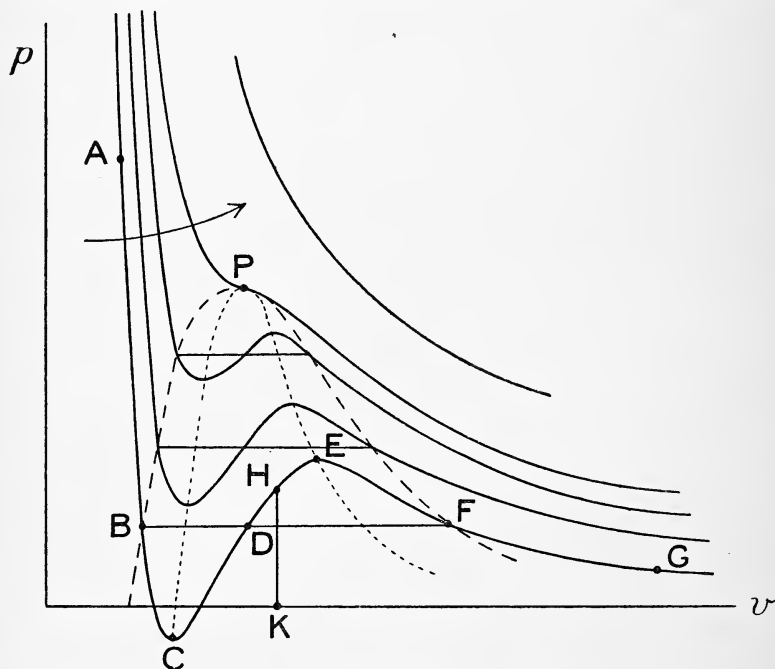
Fig. 12·51

constant temperature) for equilibrium. The actual pressure is therefore operating so as to remove the substance more and more from the state of equilibrium. It should be observed that in the deduction of van der Waals' equation, the whole of the substance is supposed to be in the *same* state at the same instant. Suppose it were possible for the whole of the substance to be in the state, H, at some instant. A slight local increment in pressure beyond HK, which is necessary for equilibrium, would result in that part of the substance changing to the condition corresponding to some point on ABC. Similarly a local diminu-

tion in pressure, however slight, would result in the substance in that locality changing to the condition represented by some point on EFG. Even supposing therefore the possibility of the whole of the substance being momentarily in the state represented by H, it would immediately break up into two states (liquid and vapour). The equilibrium at the boundary between the two phases is obviously independent of the relative quantities of the substance in these phases. The equilibrium pressure is therefore determined solely by the temperature. Consequently the portion of an isothermal in which liquid and vapour states coexist will be a horizontal straight line. Thermodynamical reasons will be given in a subsequent chapter (§ 17·4) which indicate that the situation of this horizontal line (BF in Fig. 12·51) is such as to make the areas BCD and DEF equal to one another. The states EF (supersaturated vapour) and BC (superheated liquid) can of course be produced experimentally. Indeed this fact led James Thomson to suggest that the isothermals have the shape indicated by ACEG (Fig. 12·51) before v. d. Waals developed his theory.

The maxima and minima of the v. d. Waals isothermals are located on a curve CPE, shown in the figure by a broken line. The isothermal passing through the summit, P, of this curve, and all those corresponding to higher temperatures, have no portions which slope upwards to the right, and we conclude that there is only one state for the range of temperatures beginning with that of the isothermal through P and extending upwards. This is in accordance with the fact, revealed by the experiments of Andrews, that it is impossible to liquefy a gas unless the temperature is reduced below a certain **critical temperature** characteristic of the particular gas,[1] and which according to the theory of v. d. Waals is the temperature corresponding to the isothermal through the point P. The term **gas state,** in its narrower sense, applies to the substance when its temperature exceeds the critical value.

Let us now pick out any isothermal, ACEG for example, and differentiate its equation with respect to v. We obtain

$$\left(\frac{dp}{dV} - \frac{2a}{V^3}\right)(V - b) + \left(p + \frac{a}{V^2}\right) = 0.$$

The maximum E, and the minimum C, therefore conform to

$$p + \frac{a}{V^2} = \frac{2a}{V^3}(V - b). \quad . \quad . \quad . \quad (12·55)$$

This must be the equation of the curve CPE. It will be noticed

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[1] This was suggested still earlier by Faraday.

that it cuts the axis, $p = 0$, at $V = 2b$ and $V = \infty$. The location of the critical point **P** is obtained by differentiating (**12·55**) and putting $\dfrac{dp}{dV} = 0$. We thus get

$$\frac{dp}{dV} - \frac{2a}{V^3} = -\frac{6a}{V^4}(V - b) + \frac{2a}{V^3}$$

and therefore, if V_c is the **critical volume,**

$$-\frac{2a}{V_c^3} = -\frac{6a}{V_c^4}(V_c - b) + \frac{2a}{V_c^3}$$

or $\qquad\qquad\qquad V_c = 3b$ . . . . . . (**12·56**)

We may find the **critical pressure** by substituting $3b$ (**12·56**) for V in equation (**12·55**). This gives

$$p_c = \frac{a}{27b^2} \cdot \quad . \quad . \quad . \quad . \quad (\textbf{12·561})$$

Finally we get an expression for the critical temperature by substituting the values (**12·56**) and (**12·561**) for the volume and pressure respectively in van der Waals' equation (**12·54**). This will be seen to give

$$T_c = \frac{8a}{27Rb} \quad . \quad . \quad . \quad (\textbf{12·562})$$

It is instructive to express the pressure, volume and temperature, in van der Waals' equation, in terms of the corresponding critical values as units. Representing them by π, ω and τ, we have

$$\pi = \frac{p}{p_c}, \; \omega = \frac{V}{V_c}, \; \tau = \frac{T}{T_c},$$

therefore

$$p = \frac{a}{27b^2}\pi, \; V = 3b\omega, \; T = \frac{8a}{27Rb}\tau.$$

On substituting in the equation of van der Waals we find

$$\left(\pi + \frac{3}{\omega^2}\right)\left(\omega - \frac{1}{3}\right) = \frac{8}{3}\tau \quad . \quad . \quad . \quad (\textbf{12·57})$$

an equation from which the constants, which distinguish one gas from another, have disappeared. The quantities π, ω and τ are termed the **reduced** pressure, volume and temperature respectively. A number of gases for which the *reduced* pressure, volume and temperature are respectively equal, i.e. for all of which the pressures, π, are equal and all of which occupy equal volumes ω at the same temperature, τ, are said to be in **corresponding states** and equations (**12·57**) express the **theorem of**

corresponding states from the point of view of the theory of van der Waals. The existence of a critical temperature, pressure and volume for gases is of course an experimental fact, and the theorem of corresponding states, in its widest sense, states that a relation

$$f(\pi, \omega, \tau) = 0$$

exists, in which f is the same function for all gases. It is very doubtful whether the theorem is accurately true ; but in the form (**12·57**) it represents at least a fair approximation to the truth.

Any horizontal line cuts an isothermal (Fig. 12·51) in *one* point or *three* points, as is otherwise obvious from the fact that for a given pressure and temperature (**12·541**) is a cubic equation for V, and has therefore one real root, or three real roots. We may regard the critical point, P, as a point where three real roots have coincident values. For this point therefore (**12·541**) becomes

$$V^3 - 3V^2V_c + 3VV_c{}^2 - V_c{}^3 = 0.$$

Hence

$$3V_c = b + \frac{RT_c}{p_c},$$

$$3V_c{}^2 = \frac{a}{p_c}$$

and

$$V_c{}^3 = \frac{ab}{p_c}.$$

These equations furnish an alternative way of arriving at the critical values (**12·56**), (**12·561**) and (**12·562**).

We shall now consider briefly the deviations from Boyle's law in the light of v. d. Waals' theory. For this purpose v. d. Waals' equation may be expressed in terms of $\eta (= pV)$, p and T. In (**12·54**) or (**12·541**) therefore we replace V by $\dfrac{\eta}{p}$ and so obtain

$$\eta^3 - (RT + bp)\eta^2 + ap\eta - abp^2 = 0 \ . \quad . \ (\textbf{12·58})$$

If we plot η against p (for constant temperatures) we get approximately horizontal straight lines (isothermals) in accordance with the approximate validity of Boyle's law. Differentiate (**12·58**) with respect to p twice over, keeping T constant, and then equate $\dfrac{d\eta}{dp}$ to 0. We thus obtain the equations

$$\eta^2 - \frac{a}{b}\eta + 2ap = 0 \quad . \quad . \quad . \quad (\textbf{12·581})$$

and

$$\frac{d^2\eta}{dp^2}[3\eta^2 - 2(RT + bp)\eta + ap] = 2ab \qquad (\textbf{12·582})$$

The former of these equations gives the positions of the minima (or maxima, if they are maxima). They are seen to lie on a parabola ((**12·581**), represented by the broken line in Fig. 12·52). The latter equation shows that the corresponding values of $\dfrac{d^2\eta}{dp^2}$ are positive (if we assume a to be positive ; i.e. if we suppose the intermolecular forces to be attractive), as we easily find by ignoring bp and ap, since an approximate estimate will suffice in order to find the sign of $\dfrac{d^2\eta}{dp^2}$. Thus

$$\frac{d^2\eta}{dp^2} = \frac{2ab}{3\eta^2 - 2RT\eta},$$

or
$$\frac{d^2\eta}{dp^2} = \frac{2ab}{R^2T^2},$$

since $\eta = RT$ approximately ; and its positive character is obvious. Consequently the values of $\eta(=pV)$ on the locus (**12·581**) are minima.
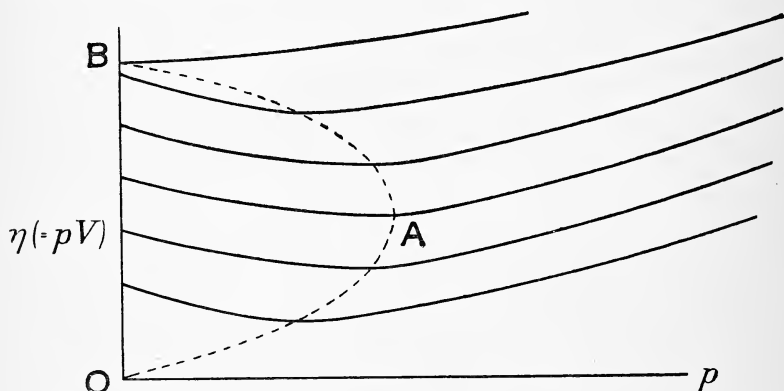
Fig. 12·52

Qualitatively the agreement between v. d. Waals' theory and the observed deviations of Boyle's law is very good. The minima are in fact observed at low temperatures to move in the direction of increasing p as we pass to higher temperatures (see the minima below A in Fig. 12·52) ; while at higher temperatures they behave in the contrary way.

On differentiating (**12·581**) with respect to η, and making $\dfrac{dp}{d\eta} = 0$, we find $\eta = \dfrac{a}{2b}$, which is the value of η for the point A (Fig. 12·52).

At the point B we find (by making $p = 0$ in (12·581)) $\eta = \frac{a}{b}$. By substitution in (12·581) or in v. d. Waals' equation, and remembering that $\eta = pV$, we find the corresponding values of V and T. These are given in the subjoined table :

	η	p	V	T
A	$\dfrac{a}{2b}$	$\dfrac{a}{8b^2}$	$4b$	$\dfrac{9a}{16Rb}$
B	$\dfrac{a}{b}$	0	∞	$\dfrac{a}{Rb}$

The theory of v. d. Waals is not however good enough quantitatively for these numerical values to be of importance. The extent of its failure can be shown very clearly by comparing the value the ratio

$$\frac{(pV)_{p=0}}{(pV)_{p=p_0}}$$

at the critical temperature with the observed value. Using equation (12·57), it becomes for very large volumes,

$$\pi\omega = \frac{8}{3},$$

since $\tau = 1$; and at the critical point $\pi = 1$ and $\omega = 1$; hence

$$\pi\omega = 1.$$

The ratio is consequently

$$2\tfrac{2}{3} \ ;$$

whereas the observed value is found to be in the neighbourhood of 3·75.[1]

§ 12·6. LOSCHMIDT'S NUMBER

It is usual to speak of the number of molecules in a gram molecule of a gas as **Loschmidt's** or **Avogadro's number**. It was first estimated by Loschmidt in 1865. The terms **atomic weight** and **molecular weight** [2] were introduced by chemists, at a time when the *absolute* masses of atoms and molecules were not yet known, to represent the masses of atoms and molecules in terms of the mass of a hydrogen atom as a unit. The atomic weight of hydrogen was therefore originally unity, and its molecular weight was taken to be 2 on the ground of chemical evidence interpreted in the light of Avogadro's hypothesis. For example the combining volumes of hydrogen and oxygen are in the ratio

[1] For an account of various alternative gas equations of Clausius, Dieterici, Callendar and others, see Ferguson, *Mechanical Properties of Fluids*. (Blackie & Sons.)

[2] 'Atomic *weight*' and 'molecular *weight*' have the sanction of long established custom ; but quite obviously 'atomic *mass*' and 'molecular *mass*' are the appropriate terms.

of two to one, and the volume of the water vapour produced is found to be the same as that of the hydrogen, when measured under like conditions of temperature and pressure. Now assuming Avogadro's hypothesis, we have in the unit volumes of hydrogen, oxygen and water vapour (at the same temperature and pressure) equal numbers of molecules, say n. Therefore the reaction may be represented in the following way :

$$2nM_H + nM_0 = 2nM_{WATER}$$

where M_H, M_0 and M_{WATER} represent the molecules of hydrogen, oxygen and water (in water vapour) respectively. Consequently

$$2M_H + M_0 = 2M_{WATER},$$

i.e. two molecules of hydrogen and one of oxygen produce two molecules of water. The simplest constitution of water consistent with the chemical evidence is H_2O. Therefore

$$2M_H + M_0 = 2H_2O,$$

and consequently
$$M_H = H_2$$
$$M_0 = O_2.$$

Later this assumed constitution for hydrogen and oxygen was confirmed by physical observations, for example by determinations of the ratio of the specific heats at constant pressure and constant volume. A gram molecule of any substance is by definition a quantity of the substance the mass of which in grams is equal to its molecular weight. More recently atomic and molecular weights have been readjusted on the basis of $O = 16$. This makes $H = 1.008$.[1]

The kinetic theory furnishes us with a means of estimating the absolute mass of a molecule, or, what amounts to the same thing, the number of molecules in a gram molecule. For this purpose we may use the following equations :

$$\overline{c^2} = \frac{3RT}{M}, \text{ equivalent to } (11\cdot911)$$

$$\mu = \frac{m\bar{c}}{4\pi\sigma^2} \quad . \quad . \quad . \quad . \quad (12\cdot322)$$

$$v = \frac{b}{4} = N\frac{\pi\sigma^3}{6},$$

$$M = Nm,$$

and we might add

$$\overline{c^2} = \frac{3\pi}{8}(\bar{c})^2 \quad . \quad . \quad . \quad . \quad (12\cdot172)$$

[1] Quite recent experimental investigations of the relative masses of the atoms of isotopes have led to a further very minute readjustment.

but some of the assumptions underlying these formulae, for instance that of spherical molecules, are so rough that we may just as well assume $\overline{c^2} = (\bar{c})^2$. The symbols have the meanings :

   $M \equiv$ mass of a gram molecule ;
   $R \equiv$ gas constant for a gram molecule ;
   $c \equiv$ velocity of a molecule, the bar indicating averages ;
   $\mu \equiv$ viscosity ;
   $m \equiv$ mass of a molecule ;
   $v \equiv$ total proper volume of the molecules ;
   $b \equiv$ v. d. Waals' constant ;
   $N \equiv$ the number of molecules in a gram molecule (Loschmidt's number).

We have in these four equations four unknown quantities, namely $\bar{c}$, σ, m and N ; the other quantities being given by experimental observations. As an illustration let us take the case of hydrogen.

   $R = 8·315 \times 10^7$ ergs per $°$C. (the same approximately for all gases).
   $M = 2·016$ gram.
   $T = 273$, if we chose the temperature of melting ice.
   $\mu = 86 \times 10^{-6}$ gram per cm. per sec.
   $b = 19·7$ c.c. for a gram molecule and hence $v = 4·925$.

When we substitute these data we find

$$\sigma = 2·74 \times 10^{-8} \text{ cm.,}$$
$$N = 4·6 \times 10^{23}.$$

Obviously these numbers cannot be regarded as expressing anything better than the order of magnitude of σ and N.

§ 12·7.   BROWNIAN MOVEMENT

In 1827 the botanist Robert Brown observed that the pollen grains of **clarkia pulchella**, when suspended in water, were in a constant state of agitation. Further investigation has shown that the phenomenon is not peculiar to pollen grains, and is not confined to particles which are living organisms. It can in fact be observed with small particles of any kind suspended in a liquid or gas. It is independent of the chemical constitution of the particles and is not due to external vibrations, or to motions in the suspending fluid due to temperature inequalities. When every precaution has been taken to get rid of such disturbances it still persists. In the words of Perrin, ' *Il est éternel et spontané.*' These characteristics of the Brownian movement led Christian Wiener in 1863 to the conclusion that it was due to the motion

of agitation of the molecules of the suspending medium. The movement is more violent in the case of small particles than in the case of larger ones as Brown himself observed ; a fact which supports the conclusion that Wiener arrived at.

§ 12·8. Osmotic Pressure of Suspended Particles

Imagine a large number of small particles, all having the same mass, suspended in a fluid of smaller density. Let n be the number of them per unit volume at a height z from the base of the containing vessel when statistical equilibrium has been attained, and m' the excess of the mass of a single particle over that of an equal volume of the suspending fluid. If p be the (osmotic) pressure due to the particles, we have by (**10·6**)

$$-\frac{dp}{dz} = nm'g,$$

or
$$dp = -nm'gdz \quad . \quad . \quad . \quad . \quad (12·8)$$

and according to the kinetic theory

$$p = nkT,$$

and therefore
$$\frac{dp}{p} = -\frac{m'g}{kT}dz,$$

or
$$\frac{dn}{n} = -\frac{m'g}{kT}dz.$$

Hence

$$\log\frac{n_0}{n} = \frac{m'g}{kT}h . \quad . \quad . \quad . \quad . \quad (12·81)$$

where n_0 is the number of particles per unit volume in a horizontal plane $z = M$, and n the number per unit volume in a plane $z = M + h$.

Perrin verified this formula experimentally by directly counting the numbers of small equal spherules of gamboge suspended at various heights in water in a small vessel which was placed under a microscope. He determined the size and mass of the spherules by various methods ; e.g. by measuring the length of a row of them and counting the number in the row ; by weighing a known or estimated number of them ; and by measuring the rate of fall of a spherule through the water and applying Stokes' law (**11·792**). The data which he thus obtained enable k to be found and hence also Loschmidt's number N, since $Nk = R$, where $R(= 8·315 \times 10^7$ ergs per ° C.) is the gas constant for a gram molecule. In this way Perrin found for N numbers varying from $6·5 \times 10^{23}$ to $7·2 \times 10^{23}$.

Perrin carried out a great variety of experiments which not only settled any question as to the nature of the Brownian movement, but constituted most important tests of the kinetic theory of gases. Only one other of these investigations will be dealt with here. It is based on a formula deduced by Einstein. The equations of motion of a single spherule may be written in the form :

$$m\frac{d^2x}{dt^2} = -S\frac{dx}{dt} + X,$$

$$m\frac{d^2y}{dt^2} = -S\frac{dy}{dt} + Y,$$

$$m\frac{d^2z}{dt^2} = -S\frac{dz}{dt} + Z ; \quad . \quad . \quad . \quad . \quad (12\cdot82)$$

where $\left(S\frac{dx}{dt}, S\frac{dy}{dt}, S\frac{dz}{dt}\right)$ represents the resistance of the fluid to the motion of the spherule, and (X, Y, Z) is the force due to bombardments by the fluid molecules. By Stokes' law (11·792)

$$S = 6\pi r\mu$$

where r is the radius of a spherule, and μ is the viscosity of the fluid. Multiplying the first of the equations (12·82) by x we have

$$mx\frac{d^2x}{dt^2} = -Sx\frac{dx}{dt} + xX,$$

and therefore

$$m\frac{d}{dt}\left(x\frac{dx}{dt}\right) - m\left(\frac{dx}{dt}\right)^2 = -\frac{S}{2}\frac{d(x^2)}{dt} + xX.$$

Consequently $\dfrac{m}{2}\dfrac{d^2}{dt^2}(x^2) - m\left(\dfrac{dx}{dt}\right)^2 = -\dfrac{S}{2}\dfrac{d(x^2)}{dt} + xX$ (12·821)

If now $\overline{x^2}$ be the average value of x^2 for a large number of the spherules, which are of course supposed to be exactly alike, we get from (12·821)

$$\frac{m}{2}\frac{d^2(\overline{x^2})}{dt^2} - kT = -\frac{S}{2}\frac{d(\overline{x^2})}{dt} \quad . \quad . \quad (12\cdot822)$$

This follows because $m\left(\dfrac{dx}{dt}\right)^2$ is two-thirds of the average kinetic energy of translation of a spherule, and as the value of X at any given place is just as likely to be positive as negative, $\overline{xX} = 0$.

If we abbreviate by writing

$$\frac{d\overline{x^2}}{dt} = \varepsilon,$$

(12·822) becomes

$$\frac{m}{2}\frac{d\varepsilon}{dt} - kT = -\frac{S}{2}\varepsilon,$$

or

$$\frac{d\varepsilon}{dt} + \frac{S}{m}\varepsilon = \frac{2kT}{m} \quad . \quad . \quad . \quad (12\cdot823)$$

This may be written

$$\frac{d}{dt}\left\{\varepsilon - \frac{2kT}{S}\right\} + \frac{S}{m}\left\{\varepsilon - \frac{2kT}{S}\right\} = 0,$$

and therefore

$$\varepsilon - \frac{2kT}{S} = Ae^{-\frac{S}{m}t} \quad . \quad . \quad . \quad (12\cdot83)$$

where A is a constant of integration. If t is sufficiently long, the right-hand side of this equation is not sensibly different from zero, and we have

$$\frac{d\overline{x^2}}{dt} = \frac{2kT}{S},$$

and consequently

$$\overline{x^2} = \frac{2kT}{S}\tau$$

or

$$\overline{x^2} = \frac{kT}{3\pi r\mu}\tau \quad . \quad . \quad . \quad . \quad (12\cdot84)$$

where $\overline{x^2}$ is the mean value of the square of the displacement in the X direction during a sufficiently long period of time τ. This is Einstein's formula.

Perrin measured $\overline{x^2}$, by means of a vertical microscope capable of motion in a horizontal plane, the individual measurements of x^2 being made on the same granule, thus eliminating the errors, due to slight differences in size, which might have resulted from observations on different granules. He thus deduced values for Loschmidt's number between $5\cdot5 \times 10^{23}$ and 8×10^{23}, his mean value being $6\cdot88 \times 10^{23}$.

The importance of these results does not lie of course in the precision of the numerical results, but in the test they furnished of the essential soundness of the kinetic theory.

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CHAPTER XI

STATISTICAL MECHANICS

§ 12·9. PHASE SPACE AND EXTENSION IN PHASE

IMAGINE a very large number of Hamiltonian systems (i.e. dynamical systems subject to the canonical equations (**8·43** and **8·46**) of Hamilton) all exactly alike and having each n degrees of freedom. For simplicity we shall suppose they do not interact with one another at all. Let the number of them which have

$$q_1 \text{ between } q_1 \text{ and } q_1 + dq_1,$$
$$q_2 \quad ,, \quad q_2 \quad ,, \quad q_2 + dq_2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$q_n \quad ,, \quad q_n \quad ,, \quad q_n + dq_n,$$
$$p_1 \quad ,, \quad p_1 \quad ,, \quad p_1 + dp_1$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$p_n \quad ,, \quad p_n \quad ,, \quad p_n + dp_n,$$

be

$$\rho \, dq_1 dq_2 \ldots dq_n dp_1 dp_2 \ldots dp_n \quad . \quad . \quad (12\text{·}9)$$

The density, ρ, may be regarded as a function of $q_1, q_2, \ldots q_n, p_1, p_2 \ldots p_n$. It will be convenient, sometimes, to replace $q_1, q_2 \ldots q_n$ by $\xi_1, \xi_2, \ldots \xi_n$ respectively and $p_1, p_2, \ldots p_n$ by $\xi_{n+1}, \xi_{n+2}, \ldots \xi_{2n}$; so that (**12·9**) may be written

$$\rho \, d\xi_1 d\xi_2 \ldots d\xi_n d\xi_{n+1} \ldots d\xi_{2n} \quad . \quad (12\text{·}901)$$

ρ being a function of $\xi_1, \xi_2, \ldots \xi_{2n}$. For illustration consider the case where each system has only one degree of freedom and consequently ρ is a function of q and the associated p. We may represent the state of the assemblage of systems on a plane, using q and p (or ξ_1 and ξ_2) as rectangular co-ordinates, and the number of systems for which q lies between q and $q + dq$, and p between p and $p + dp$ is

$$\rho \, dq \, dp,$$

or

$$\rho \, d\xi_1 d\xi_2.$$

The language and symbolism appropriate for this graphical representation of the distribution of the systems may profitably be extended to an assemblage of systems each of which has n

degrees of freedom, although we may not be able to visualize $2n$ axes of coordinates. We shall term the space of such a diagram the **representative space** or **phase space**, and ρ the **density** of the distribution of the systems in phase.

The equation of continuity (**10·52**) suggests that

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \xi_1}(\rho \dot{\xi}_1) + \frac{\partial}{\partial \xi_2}(\rho \dot{\xi}_2) + \ldots + \frac{\partial}{\partial \xi_n}(\rho \dot{\xi}_{2n}) = 0, \quad \textbf{(12·91)}$$

or

$$\frac{\partial \rho}{\partial t} + \sum \frac{\partial(\rho \dot{\xi}_s)}{\partial \xi_s} = 0.$$

This is easily established in the following way : Consider the plane (or boundary), ξ_s for example, of the elementary region included between

$$\xi_1 \text{ and } \xi_1 + d\xi_1,$$
$$\xi_2 \text{ ,, } \xi_2 + d\xi_2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\xi_{2n} \text{ ,, } \xi_{2n} + d\xi_{2n}.$$

Obviously the number of systems which cross this boundary and enter the element in the time dt is expressed by

$$\rho \dot{\xi}_s dt \, d\xi_1 d\xi_2 \ldots d\xi_{s-1} \, d\xi_{s+1} \ldots d\xi_{2n},$$

in which product the differential $d\xi_s$ is missing, its place being taken by $\dot{\xi}_s dt$. Some of these may of course cross one or more of the $2n - 1$ remaining boundaries ; but the number of them doing so will be a small quantity of still higher order, and need not be further regarded. The number of systems *leaving* the element through the boundary $\xi_s + d\xi_s$ will clearly be

$$\left\{ \rho \dot{\xi}_s + \frac{\partial(\rho \dot{\xi}_s)}{\partial \xi_s} \, d\xi_s \right\} dt \, d\xi_1 d\xi_2 \ldots d\xi_{s-1} \, d\xi_{s+1} \ldots d\xi_{2n}.$$

On subtraction we find for the excess of the number of systems leaving the element of volume of the representative space in the time dt over that entering it

$$\frac{\partial(\rho \dot{\xi}_s)}{\partial \xi_s} dt \, d\xi_1 d\xi_2 \ldots d\xi_s \ldots d\xi_{2n} \, ;$$

and when we take account of the remaining boundaries we get the result

$$\sum \frac{\partial(\rho \dot{\xi}_s)}{\partial \xi_s} dt \, d\xi_1 d\xi_2 \ldots d\xi_{2n}.$$

This must equal

$$-\frac{\partial \rho}{\partial t} dt \, d\xi_1 d\xi_2 \ldots d\xi_{2n},$$

and equation (12·91) results from equating the two expressions. Since this equation may be written in the form

$$\frac{\partial \rho}{\partial t} + \dot{\xi}_1 \frac{\partial \rho}{\partial \xi_1} + \dot{\xi}_2 \frac{\partial \rho}{\partial \xi_2} + \ldots + \dot{\xi}_{2n} \frac{\partial \rho}{\partial \xi_{2n}}$$

$$+ \rho \left(\frac{\partial \dot{\xi}_1}{\partial \xi_1} + \frac{\partial \dot{\xi}_2}{\partial \xi_2} + \ldots + \frac{\partial \dot{\xi}_{2n}}{\partial \xi_{2n}} \right) = 0,$$

we may use the symbolism of § 10·7 (see equations **10·701** and **10·702**) and write

$$\frac{D\rho}{Dt} + \rho \left(\frac{\partial \dot{\xi}_1}{\partial \xi_1} + \frac{\partial \dot{\xi}_2}{\partial \xi_2} + \ldots + \frac{\partial \dot{\xi}_{2n}}{\partial \xi_{2n}} \right) = 0 \quad . \qquad (12 \cdot 911)$$

In this equation $\frac{D\rho}{Dt}$ represents the rate of change of ρ as we follow the motion along a stream line in the representative space.

In the earlier notation (12·911) becomes

$$\frac{D\rho}{Dt} + \rho \sum_{s=1}^{s=n} \left\{ \frac{\partial \dot{q}_s}{\partial q_s} + \frac{\partial \dot{p}_s}{\partial p_s} \right\} = 0 \qquad . \quad . \qquad (12 \cdot 912)$$

Now it follows from the canonical equations

$$\dot{q}_s = \frac{\partial H}{\partial p_s},$$

$$\dot{p}_s = - \frac{\partial H}{\partial q_s},$$

(where H is the energy of any one of the systems in the element $dq_1 \ldots dq_n dp_1 \ldots dp_n$ or $d\xi_1 \ldots d\xi_{2n}$) that

$$\frac{\partial \dot{q}_s}{\partial q_s} + \frac{\partial \dot{p}_s}{\partial p_s} = 0$$

for every s. Therefore

$$\frac{D\rho}{Dt} = 0 \quad . \quad . \quad . \quad . \quad . \quad (12 \cdot 92)$$

This result is known as **Liouville's theorem.** We can express it in an alternative way. If $\triangle N$ be the number of systems in the element $\triangle \xi_1 \triangle \xi_2 \ldots \triangle \xi_{2n}$,

$$\triangle N = \rho \triangle \xi_1 \triangle \xi_2 \ldots \triangle \xi_{2n},$$

or briefly

$$\triangle N = \rho \triangle \xi ;$$

and

$$\frac{D(\triangle N)}{Dt} = \triangle \xi \frac{D\rho}{Dt} + \rho \frac{D(\triangle \xi)}{Dt}.$$

If we confine the equation to the same systems

$$\frac{D(\triangle N)}{Dt} = 0,$$

and by Liouville's theorem

$$\frac{D\varrho}{Dt} = 0, \text{ and } \varrho = \text{constant};$$

hence

$$\varrho \frac{D(\triangle\xi)}{Dt} = 0,$$

and $\triangle\xi_1 \triangle\xi_2 \ldots \triangle\xi_{2n} = \text{constant}$.

If therefore we follow the motion of N systems in the representative space, the volume,

$$\int dq_1 \ldots dq_n \, dp_1 \ldots dp_n, \quad . \quad . \quad \textbf{(12·921)}$$

which they occupy in it will remain constant. In the language of Willard Gibbs their **extension in phase** remains constant.

§ 13. Canonical Distributions

If the number of systems per unit volume at every point in the representative space is constant, i.e. if

$$\frac{\partial\varrho}{\partial t} = 0$$

everywhere, we have **statistical equilibrium.** The condition for statistical equilibrium is therefore **(12·92)**

$$\sum_{s=1}^{s=2n} \xi_s \frac{\partial\varrho}{\partial\xi_s} = 0 \quad . \quad . \quad . \quad . \quad \textbf{(13)}$$

or

$$\sum_{s=1}^{s=n} \left\{ \dot{q}_s \frac{\partial\varrho}{\partial q_s} + \dot{p}_s \frac{\partial\varrho}{\partial p_s} \right\} = 0. \quad . \quad . \quad . \quad \textbf{(13·01)}$$

This condition will be satisfied by

$$\varrho = f(H) \quad . \quad . \quad . \quad . \quad . \quad \textbf{(13·02)}$$

where f is any function, and H is the energy of a system; for if we represent $\dfrac{d\varrho}{dH}$ by f',

$$\frac{\partial\varrho}{\partial q_s} = \frac{\partial H}{\partial q_s} f' = -\dot{p}_s f',$$

and

$$\frac{\partial\varrho}{\partial p_s} = \frac{\partial H}{\partial p} f' = \dot{q}_s f',$$

and consequently

$$\dot{q}_s\frac{\partial \varrho}{\partial q_s} + \dot{p}_s\frac{\partial \varrho}{\partial p_s} = (-\dot{q}_s\dot{p}_s + \dot{p}_s\dot{q}_s)f' = 0.$$

The particular case

$$\rho = Ae^{-\frac{H}{\Theta}} \quad . \quad . \quad . \quad . \quad . \quad (13\cdot03)$$

where Θ and A are constants, is of great interest, and is naturally suggested as a generalization of the Maxwellian law of distribution, § 12·1. The constant A can be expressed in terms of Θ and the inherent constants of the individual systems constituting the assemblage by substituting the expression (13·03) for ρ in

$$\int \cdots \int \rho d\xi_1 \cdots d\xi_{2n} = N \quad . \quad . \quad (13\cdot04)$$

where N is the total number of systems in the assemblage. A distribution defined by (13·03) is called by Willard Gibbs a **canonical distribution,** the constant, Θ, being the **modulus** of the distribution.

§ 13·1. Statistical Equilibrium of Mutually Interacting Systems

We have been studying a type of assemblage, the individual members of which are conservative systems, and do not interact with one another at all, and in which therefore the energy is distributed in such a way that a definite portion of it is assigned to each system. No actual assemblage can be strictly of this type. There must be some interaction between individual systems, and consequently some of the energy must be the mutual energy of groups of two or more systems. In what follows we shall take this interaction into account ; but we shall restrict our attention to cases where the mutual energy is a negligible fraction of the whole energy of the assemblage. Let the total number of systems forming the assemblage be N, and imagine the phase space to be divided into very small and equal elements $\triangle\omega_1, \triangle\omega_2, \triangle\omega_3 \ldots$. We may denote the number of systems in the elements $\triangle\omega_1, \triangle\omega_2, \triangle\omega_3 \ldots$ by $N_1, N_2, N_3, \ldots$ respectively and the energy of each system in these elements by $E_1, E_2, E_3, \ldots$ respectively, the total energy being E. We have therefore

$$N = \sum_{s=1, 2, 3, \ldots} N_s,$$

$$E = \sum_{s=1, 2, 3, \ldots} E_sN_s \quad . \quad . \quad . \quad . \quad (13\cdot1)$$

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It is convenient to write

$$f_s = \frac{N_s}{N},$$

and $\bar{E}$ for the average energy of a system, so that equations (13·1) become

$$1 = \sum_{s=1, 2, 3, \ldots} f_s,$$

$$\bar{E} = \sum_{s=1, 2, 3, \ldots} E_s f_s \quad . \quad . \quad . \quad . \quad (13·101)$$

Among the various distributions of the systems in the phase space, the only one which can be permanently in statistical equilibrium is that which has the greatest probability. In order to find a starting point for attacking the problem of determining the relative probabilities of different distributions, let us consider the following illustration : Imagine a large number of similar balls to be projected, so that they fall into one or other of three receptacles, A, B and C. It may happen that they distribute themselves equally among the three receptacles, and hence the probability that any one of the balls is in the receptacle A is the same as the probability of its being in B, or in C. This is often expressed in the form : the *a priori* probability that a given ball is in the receptacle A is the same as the *a priori* probability of its being in B, or in C. The term *a priori* is used because the probability in question is one of the premises from which we start out when we wish to find the probability of a given distribution of some definite number of balls in the three receptacles ; e.g. a total number of 6 balls, 3 in A, 2 in B and 1 in C.

If in the projection of the balls, one of the receptacles is favoured in some way, so that when a large number of them is projected, twice as many fall into B as into either A or C, the *a priori* probabilities of a particular ball being in *A*, *B* or *C* are as 1 : 2 : 1. In the former case A, B and C are said to have equal **weights,** in the latter their weights are 1, 2 and 1 respectively. If the weights (or *a priori* probabilities) associated with the receptacles are all equal, the probability of a given distribution among them of a definite number of balls is equal to the number of ways (or complexions) in which this distribution can be made, divided by the sum of the numbers of complexions of all possible distributions. Taking the example of two receptacles A and B and a distribution in which 4 balls are in A and 2 in B, out of a total of 6 balls ; the number of complexions is

$$\frac{6!}{4!\,2!},$$

while the sum of the numbers of complexions of all possible distributions of 6 balls between the two receptacles is

$$1 + \frac{6}{1} + \frac{6.5}{1.2} + \frac{6.5.4}{1.2.3} + \frac{6.5.4.3}{1.2.3.4} + \frac{6.5.4.3.2}{1.2.3.4.5} + 1.$$

This is the sum of the coefficients in the expansion of

$$(a + b)^6,$$

and is therefore equal to

$$2^6.$$

Hence the probability required is

$$\frac{6!}{4! \, 2!} 2^{-6}.$$

More generally if N be the total number of balls, distributed among n receptacles, so that there are N_1, N_2, N_3, ... N_n balls respectively in the receptacles 1, 2, 3, ... n; the probability of the distribution will be

$$\frac{N!}{N_1! \, N_2! \, N_3! \, \ldots \, N_n!} n^{-N} \qquad . \quad . \quad . \quad (13 \cdot 11)$$

In these examples we have tacitly adopted the usual convention that certainty is represented by unity. It is more convenient however for the purposes we have in view to use the total number—n^N in (13·11)—of the complexions of all the possible distributions, as representing certainty; in which case (13·11) is replaced by

$$\frac{N!}{N_1! \, N_2! \, N_3! \, \ldots \, N_n!} \qquad . \quad . \quad . \quad . \quad (13 \cdot 12)$$

Adopting this convention, and assuming that the *a priori* probabilities associated with all the elements $\triangle \omega_1$, $\triangle \omega_2$, $\triangle \omega_3$, ... of the phase space are equal; the probability, P, of the distribution in which

N_1 systems are in the element $\triangle \omega_1$,
N_2 ,,   ,, ,, ,,   ,, $\triangle \omega_2$,
N_3 ,,   ,, ,, ,,   ,, $\triangle \omega_3$,

and so on, is clearly

$$P = \frac{N!}{N_1! \, N_2! \, N_3! \, \ldots} \qquad . \quad . \quad . \quad (13 \cdot 13)$$

We assume N_1, N_2, N_3, ... to be individually very large numbers, and we may in consequence make use of Stirling's theorem, namely

$$n! = \sqrt{2\pi n} \; e^{-n} n^n, \quad . \quad . \quad . \quad . \quad (13 \cdot 14)$$

where n is a large integer, strictly speaking an infinite integer.

It follows that
$$\log n! = n \log n \quad . \quad . \quad . \quad . \quad . \quad . \quad (13 \cdot 141)$$
and hence
$$\log P = N \log N - \sum_{s=1, 2, 3, \ldots} N_s \log N_s,$$
or
$$\psi = \log P = - N \sum_{s=1, 2, 3, \ldots} f_s \log f_s \quad . \quad . \quad . \quad (13 \cdot 15)$$

The most probable distribution is that for which P, and consequently ψ, has the biggest value, subject to the conditions (13·101). The maximum value of ψ is therefore determined by
$$\delta\psi = - N \sum_{s=1, 2, 3, \ldots} (\log f_s + 1)\delta f_s = 0,$$
the δf_s being subject to the limitations imposed by
$$\sum_{s=1, 2, 3, \ldots} \delta f_s = 0,$$
and
$$\delta\overline{E} = \sum_{s=1, 2, 3, \ldots} E_s \delta f_s = 0,$$
which merely express the fact that the total number of systems, and the total energy remain constant. These equations are equivalent to
$$\sum_{s=1, 2, 3, \ldots} \log f_s \delta f_s = 0,$$
$$\sum_{s=1, 2, 3, \ldots} E_s \delta f_s = 0,$$
$$\sum_{s=1, 2, 3, \ldots} \delta f_s = 0. \quad . \quad . \quad . \quad . \quad (13 \cdot 16)$$

Hence it follows that the most probable distribution is given by
$$\log f_s + \beta E_s + a = 0 \quad . \quad . \quad . \quad . \quad (13 \cdot 17)$$
where a and β are constants, and consequently
$$f_s = Be^{-\beta E_s} \quad . \quad . \quad . \quad . \quad (13 \cdot 171)$$
in which B is a constant. This is identical with the canonical distribution already described, since B can be put in the form
$$B = A \triangle \omega,$$
or
$$B = A dq_1 dq_2 \ldots dq_n dp_1 dp_2 \ldots dp_n \quad . \quad (13 \cdot 172)$$
where A is a constant, and hence
$$\beta = \frac{1}{\Theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13 \cdot 18)$$

The constant B can of course be expressed in terms of β (or Θ), since

$$\Sigma f_s = 1 = B\Sigma e^{-\beta E_s} \quad . \quad . \quad . \quad (13 \cdot 185)$$

The maximum value of ψ is obtained by substituting the expression (13·171) for f_s in (13·15). We thus have

$$\psi_m = - N \sum_{s=1, 2, 3, \ldots} B e^{-\beta E_s^a} (\log B - \beta E_s),$$

or

$$\psi_m = - N \log B + \beta E \; . \quad . \quad . \quad . \quad . \quad (13 \cdot 19)$$

in consequence of (13·185) and (13·101).

§ 13·2. CRITERIA OF MAXIMA AND MINIMA

We have tacitly assumed that ψ_m is a maximum; but the foregoing argument does not distinguish between a maximum and a minimum. To settle this question we expand $\delta\psi$, the small increment of ψ due to small increments δf_s. Since

$$\psi = - N\Sigma f_s \log f_s,$$

we have

$$\delta\psi = - N\Sigma\Big[(f_s + \delta f_s) \log (f_s + \delta f_s) - f_s \log f_s\Big],$$

or

$$\delta\psi = - N\Sigma\Big[(f_s + \delta f_s)\Big\{\log f_s + \log \Big(1 + \frac{\delta f_s}{f_s}\Big)\Big\} - f_s \log f_s\Big].$$

Therefore

$$\delta\psi = - N\Sigma\Big[\delta f_s \log f_s + (f_s + \delta f_s) \log \Big(1 + \frac{\delta f_s}{f_s}\Big)\Big],$$

$$= - N\Sigma\Big[\delta f_s \log f_s + f_s\Big\{\tfrac{1}{2}\Big(\frac{\delta f_s}{f_s}\Big)^2 - \tfrac{1}{6}\Big(\frac{\delta f_s}{f_s}\Big)^3 + - \ldots\Big\}\Big].$$

Now when

$$\delta\psi = \psi - \psi_m,$$

this reduces to

$$\delta\psi = - N\Sigma f_s\Big\{\tfrac{1}{2}\Big(\frac{\delta f_s}{f_s}\Big)^2 - \tfrac{1}{6}\Big(\frac{\delta f_s}{f_s}\Big)^3 + - \ldots\Big\}$$

which is essentially *negative* whatever the δf_s may be, provided they are small enough. Hence ψ_m is a maximum.

§ 13·3. SIGNIFICANCE OF THE MODULUS

Let us now consider a small increment $d\psi_m$ *due to a small change dE in the energy of the whole assemblage.* The values of B and β, which for a given value of E are constants, will now experience increments dB and $d\beta$, and we have from (13·19)

$$d\psi_m = - N\frac{dB}{B} + \beta dE + E d\beta \quad . \quad . \quad (13 \cdot 30)$$

Differentiating (13·185) we find

$$0 = dB \sum_{s=1, 2, 3, \ldots} e^{-\beta E_s} - B \sum_{s=1, 2, 3, \ldots} E_s e^{-\beta E_s} d\beta$$

or

$$0 = \frac{dB}{B} - \frac{E}{N} d\beta$$

and consequently, on substituting in (13·30),

$$d\psi_m = \beta dE,$$

or

$$\boxed{d\psi_m = \frac{dE}{\Theta}} \quad \ldots \quad \ldots \quad (13\cdot31)$$

The $d\psi_m$ in this equation must be sharply distinguished from $\delta\psi$. The former represents the small increment of ψ corresponding to the increment dE of the energy of the assemblage when statistical equilibrium is practically established. The latter means a small change in ψ occurring while E remains constant, and it can only differ from zero so long as statistical equilibrium (or, strictly speaking, the most probable state) has not been reached.

We now turn to the problem of the statistical equilibrium of two assemblages, which can interchange energy with one another, but are otherwise isolated; i.e. their combined energy is a constant quantity. We shall distinguish them by the letters A and B; so that

$$E = E_A + E_B.$$

It is easy to see that

$$P = P_A . P_B,$$

where P is the probability of a state of the combined assemblages, while P_A and P_B are the probabilities of the associated states of the individual assemblages A and B respectively. Consequently

$$\psi = \psi_A + \psi_B.$$

The condition for statistical equilibrium of the combined systems is

$$\delta\psi = 0,$$

subject to

$$\delta E = 0 \quad \ldots \quad \ldots \quad \ldots \quad (13\cdot32)$$

Now since the individual systems, A and B, are themselves in statistical equilibrium any small changes in ψ_A or ψ_B must be due to transfer of energy from A to B or B to A, and are therefore properly represented by $d\psi_A$ and $d\psi_B$. Consequently

$$\delta\psi = d\psi_A + d\psi_B,$$

and

$$\delta E = dE_A + dE_B . \quad \ldots \quad \ldots \quad (13\cdot33)$$

The conditions for statistical equilibrium are therefore
$$d\psi_A + d\psi_B = 0,$$
$$dE_A + dE_B = 0 ; \quad \cdots \cdots \quad (13\cdot34)$$
and, by (13·31),

$$d\psi_A = \frac{dE_A}{\Theta_A},$$

$$d\psi_B = \frac{dE_B}{\Theta_B}.$$

On substituting for $d\psi_A$ and $d\psi_B$ in equations (13·34) we get
$$\frac{dE_A}{\Theta_A} - \frac{dE_A}{\Theta_B} = 0 ;$$
whence

$$\Theta_A = \Theta_B \quad \cdots \cdots \cdots \quad (13\cdot35)$$

This then is the condition that the two assemblages may be in statistical equilibrium with one another.

Any interaction between two assemblages which have not yet reached statistical equilibrium must be such that $\delta\psi$ or

$$\frac{dE_A}{\Theta_A} + \frac{dE_B}{\Theta_B} \quad \cdots \cdots \quad (13\cdot36)$$

is a positive quantity, because it is bound to have such a character as to bring about a condition which is more probable.[1]   Therefore

$$\frac{dE_A}{\Theta_A} - \frac{dE_A}{\Theta_B} \text{ is positive.}$$

If now
$$\Theta_A > \Theta_B$$
dE_A must be negative ; i.e. energy must flow from the assemblage which has the greater modulus, Θ.

It is now clear that Θ plays the part of temperature, and we have reached the stage when we may claim to have given an explanation of the more obvious features of thermal phenomena in mechanical terms.   Reference to §§ 12 and 12·1, and more especially to equations (12), (12·01) and (12·16) will indicate that we must identify Θ with kT.   For the thermal equilibrium of two assemblages (two gases for example)
$$\Theta_A = \Theta_B, \text{ by } (13\cdot35),$$
and the physical meaning of temperature necessitates that
$$T_A = T_B,$$

[1] Strictly speaking, we may only equate $d\psi_A$ to $\dfrac{dE_A}{\theta_A}$ when the assemblage A is itself in statistical equilibrium, so that the expression (13·36) may only be employed for $\delta\psi$ when statistical equilibrium has nearly been attained.   It will however suffice for the present purpose if we suppose that this is the case.

consequently
$$\frac{\Theta_A}{T_A} = \frac{\Theta_B}{T_B},$$

or
$$k_A = k_B,$$

and the assumption of the universal character of the constant a (or $\frac{3}{2}k$) in equation (12) is now justified.

§ 13·4. ENTROPY

In the chapters on thermodynamics we shall meet with a quantity, ϕ, first introduced by Clausius and known as **entropy.** We shall see that when a system is nearly in thermal equilibrium

$$d\phi = \frac{dQ}{T},$$

where dQ is the heat communicated to the system and $d\phi$ is the corresponding increase in its entropy.   If we compare this relation with

$$d\psi = \frac{dE}{\Theta},$$

we see at once that

$$k\psi = \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (13\cdot4)$$

In consequence of this relationship k is often called the **entropy constant.**   It is also known as **Boltzmann's constant.**

§ 13·5. THE THEOREM OF EQUIPARTITION OF ENERGY

The general expression (8·26) for the kinetic energy of a Hamiltonian system simplifies in many cases to a sum involving squares of momenta, but not their products.   When this happens, the energy of the system takes the form

$$E = V + a_1 p_1{}^2 + a_2 p_2{}^2 + \ldots + a_s p_s{}^2 + \ldots \quad (13\cdot5)$$

where $a_1, a_2, \ldots a_s, \ldots$ are either constants or functions of the q's only.   Examples are : a particle, a rigid body or also a system consisting of two mutually gravitating bodies.   It is convenient to term $a_1 p_1{}^2$, $a_2 p_2{}^2$, $\ldots a_s p_s{}^2$, $\ldots$ etc., the kinetic energies associated with the co-ordinates 1, 2, $\ldots$ s, $\ldots$ etc., respectively.   We can now establish that, in any assemblage of this kind, the average kinetic energy (of a system) associated with any co-ordinate, s, is the same for all the co-ordinates and equal to $\frac{\Theta}{2}$ or to $\frac{kT}{2}$.   The number of systems in

the element $dq_1 dq_2 \ldots dq_n dp_1 dp_2 \ldots dp_n$ may be expressed in the form :

$$NAe^{-\dfrac{V + a_1 p_1{}^2 + a_2 p_2{}^2 + \ldots + a_s p_s{}^2 + \ldots}{\Theta}} dq_1 dq_2 \ldots dq_n dp_1 \ldots dp_n,$$

where N is the total number of systems in the assemblage. The total kinetic energy associated with the co-ordinate, s, in this element of the phase space is

$$NAa_s p_s{}^2 e^{-\dfrac{V + a_1 p_1{}^2 + a_2 p_2{}^2 + \ldots + a_s p_s{}^2 + \ldots}{\Theta}} dq_1 dq_2 \ldots dq_n dp_1 \ldots dp_n.$$

The average kinetic energy (in an element of volume $dq_1 dq_2 \ldots dq_n$) associated with s is consequently

$$\frac{dq_1 dq_2 \ldots dq_n \displaystyle\int a_s p_s{}^2 e^{-\dfrac{a_s p_s{}^2}{\Theta}} dp_s \displaystyle\int \ldots}{dq_1 dq_2 \ldots dq_n \displaystyle\int e^{-\dfrac{a_s p_s{}^2}{\Theta}} dp_s \displaystyle\int \ldots} \qquad (13{\cdot}51)$$

or

$$\frac{a_s \displaystyle\int p_s{}^2 e^{-\dfrac{a_s p_s{}^2}{\Theta}} dp_s}{\displaystyle\int e^{-\dfrac{a_s p_s{}^2}{\Theta}} dp_s} \qquad \ldots \qquad (13{\cdot}511)$$

Both integrals in this expression have of course the same limits —p_s may range from 0 to $+\infty$ or from $-\infty$ to $+\infty$ —in either case we get from $(12{\cdot}131)$ for the average kinetic energy

$$\tfrac{1}{2}\Theta,$$

as stated above. This is the theorem of **equipartition of energy** on which the proofs of the laws of Avogadro and Charles in § 12 were based.

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    See also references at the end of the preceding chapter.

CHAPTER XII

THERMODYNAMICS. FIRST LAW

§ 15. Origin of Thermodynamics

THERMODYNAMICS, as we understand the term, owes its origin to the Frenchman Sadi Carnot who published in 1824 a treatise entitled ' Réflexions sur la Puissance Motrice du Feu et sur les Machines propres a developper cette Puissance.' This work, one of the most important and remarkable in the whole range of physical science, was entirely ignored for more than twenty years, when its merits were recognized by Sir William Thomson, afterwards Lord Kelvin. Classical thermodynamics is based on two main principles, the first and second laws of thermodynamics. The first law, which is simply the principle of conservation of energy as applied to thermal phenomena, is commonly ascribed to Julius Robert Mayer, who, in 1842, evaluated the so-called mechanical equivalent of heat from the values of the specific heats of air at constant pressure and constant volume. In justice to Carnot it should be said that a precise and clear statement of the first law was found, after his death, in the manuscript notes which he left, and also a calculation of the mechanical equivalent of heat. The value which he found was 0·37 kilogram-metres per gram-calorie. The second law was also discovered by Carnot, and is contained in the treatise mentioned above.

While classical thermodynamics is based on the two laws already mentioned, a ' third law of thermodynamics ' has been added in recent times by the German physical chemist, W. Nernst.

§ 15·1. Temperature

We may define the term ' temperature of a body ' in a rough way as its hotness expressed on a numerical scale. The term ' hotness ' has reference to the sensation we experience in touching a hot body. Such sensations do not enable us to construct a scale of temperature with precision, and we have therefore to make use of appropriate physical quantities for this purpose.

Of these physical quantities, one which is very commonly used is the volume of a fixed quantity of some liquid, usually mercury. We assume that the reader is familiar with the mercury thermometer.   An arbitrary scale, for example a millimetre scale, marked on the stem of such a thermometer defines a scale of temperature as far as the divisions extend. If we place the thermometer in water contained in a beaker, the mercury will expand, or contract, according as it happens to be initially colder or hotter than the water, until a state of equilibrium (thermal equilibrium) is established, when the top of the mercury column is at some definite mark on the arbitrary scale. If we make the water progressively hotter (in the sense that it actually feels hotter), we find as an experimental fact, that the mercury column rises in the stem of the thermometer. Another important fact of experience is the following : if we place two bodies, having very different temperatures, in contact ; for example if we surround some hot liquid contained in a copper vessel by cold water contained in a larger beaker, we find that ultimately a state of thermal equilibrium is set up, in which both the liquid in the copper vessel and the surrounding water have the same temperature. This is the case whether we judge the temperature by the sensations experienced on immersing the hand in the liquids or by noting the position of the top of the mercury column on the stem of the thermometer. We see that the readings of a mercury thermometer follow, as far as we can judge, the much rougher indications of our sensations of warmth or coldness. We may continue to adhere to the definition of temperature given above, namely, ' the hotness of a body expressed on a numerical scale ' provided that the numerical scale is defined by some physical quantity, as for example the volume of a definite quantity of mercury in thermal equilibrium with the body, the temperature of which is being expressed.

There are many other physical quantities which may be employed for defining scales of temperature and for temperature measurement, e.g. the electrical resistance of a piece of platinum wire, or the electromotive force in a thermo-couple; but whatever physical quantity be used, it must express the temperature in a way that is unambiguous over the range of temperatures that are being measured.   A water thermometer, for example, would not be a suitable instrument for temperatures immediately above that of melting ice, since, as it is gradually heated up the liquid column *descends* at first, reaches a minimum position, and then rises ; so that there are definite positions on the stem of such a thermometer each of which corresponds to *two different* temperatures.

§ 15·15.   SCALES OF TEMPERATURE.

It is usual to subject scales of temperature to the condition that the difference in temperature of a mixture of ice and water in equilibrium under the normal pressure, and saturated water vapour under the normal pressure shall be 100.[1]   These two temperatures have been found to be invariable.   This means of course—taking the case of ice and water in equilibrium under normal pressure for instance—that the indication of the thermometric device, whether it functions in terms of the volume of a definite mass of liquid, the resistance of a piece of platinum wire or in any other way, is always the same, once thermal equilibrium with the mixture has been established.

If some physical quantity x, which may be the volume of a definite quantity of mercury, the pressure of a definite quantity of some gas at constant volume, the electrical resistance of a piece of platinum wire, or any other appropriate quantity, is used for thermometric purposes and if x_0 and x_1 represent the values corresponding to the temperature of the ice and water under normal pressure (melting ice) and the saturated steam respectively, then $x_1 - x_0$ represents a difference of 100°.   A difference of 1° is defined by

$$\frac{x_1 - x_0}{100}.$$

In the case of the Centigrade scale the temperature of the melting ice is marked 0°, and on this scale the value x would therefore represent the temperature,

$$t = (x - x_0) \div \frac{(x_1 - x_0)}{100},$$

or

$$t = 100 \, \frac{x - x_0}{x_1 - x_0} \quad . \quad . \quad . \quad . \quad . \quad (15\cdot15)$$

It is important to notice that different physical properties x define *different* scales of temperature.   The readings of a gas thermometer for example do not agree with those of a platinum resistance thermometer.   We shall see later that the second Law of Thermodynamics provides us with a means of defining scales of temperature which are independent of the physical property used in the experimental measurement.   Meanwhile it may be noted that the product of the pressure and volume of a definite quantity of any gas is very nearly constant if the temperature (as indicated by a mercury thermometer for instance) is kept constant, i.e. the product is independent of the individual values of p or v (Boyle's law).   The product pv is a quantity which

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[1] This is merely the definition of an arbitrary unit of temperature.

increases continuously as the gas is heated and therefore is suitable for defining a scale of temperature, and it has the special merit, that it is the same function of the temperature (whatever arbitrary scale we may have adopted) for all gases, at any rate approximately (**law of Charles**). This means that if we take fixed quantities of different gases, such that pv has the same value for all of them at 0° C., it will have approximately the same value for all of them at any other temperature (§ 12). We have therefore

$$pv = f(t) \quad . \quad . \quad . \quad . \quad . \quad (15\text{·}16)$$

where t is the temperature on some definite but arbitrary scale, and f is the same function, approximately, for all gases. It is found that all gases approximate more and more closely in their behaviour to the laws of Boyle and Charles as their temperatures are raised, provided that the pressure is not unduly raised. We use the term **perfect gas,** or **ideal gas** for a hypothetical body which obeys these laws exactly and has certain other properties, to be detailed later, which actual gases approach under the conditions just mentioned. These facts suggest the use of a perfect gas to define a scale of temperature. The Centigrade **gas scale** would then be expressed by the formula

$$t = 100\frac{(pv)_t - (pv)_0}{(pv)_1 - (pv)_0} \quad . \quad . \quad . \quad . \quad (15\text{·}17)$$

It is more convenient to define a gas scale by giving equation (**15·16**) the form

$$pv = RT' \quad . \quad . \quad . \quad . \quad (15\text{·}171)$$

where t has been replaced by T' and R is a constant, the value of which is chosen so that

$$(pv)_1 - (pv)_0 = 100R.$$

The zero of temperature on this scale is called the **absolute zero,** and the constant R is the **gas constant.**

§ 15·2.   Equations of State

The equation connecting the pressure, volume and temperature of a definite mass of any substance is called its **equation of state.** The statements (**15·16**) and (**15·171**) are appropriate equations of state for an ideal gas. Other equations have been proposed, to which the behaviour of actual gases conforms more closely, for example the equation of van der Waals,

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT' \quad . \quad . \quad . \quad (15\text{·}2)$$

where a, b and R are constants characteristic of the particular gas.

§ 15·3. THERMODYNAMIC DIAGRAMS

It is convenient to represent the relation between the pressure and volume of a substance, or between the pressure and temperature, or any other pair of variables, graphically. The most important of these diagrams is that representing the relation between pressure and volume. These relations are determined by the equation of state of the substance, and the conditions to which it is subjected. For example if we take hydrogen gas, the equation of state of which is fairly accurately expressed by (15·171), and subject it to the condition of constant temperature, the graphical representation of the relation between p and v will be a rectangular hyperbola (see Fig. 15·3). It should be noted that when we speak of the pressure of a substance we mean the pressure measured *while it is in equilibrium*. This is the sense in which the term pressure is used in the equation of state.
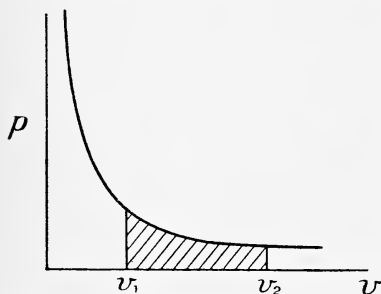
It is very important to remember that when a gas or vapour is expanding rapidly, for example, in a cylinder closed by a piston, the actual pressure exerted on the walls of the cylinder or on the piston will differ from that which would be exerted if the gas or vapour were in equilibrium, e.g. if the piston were not in motion, or if it were moving very slowly. In what follows, the term ' pressure ' will, unless the contrary is expressly stated, always be used to mean the pressure measured under conditions in which the substance is in equilibrium or expanding with extreme slowness. Any process which takes place under conditions which differ only slightly (infinitesimally) from those of equilibrium is termed a **reversible process**. Such a process is in fact *reversible* in the literal sense of the term. If for example a gas were expanding in the way mentioned above, the process differing only infinitesimally from a succession of states of equilibrium, it is obvious from the equation of state that an infinitesimal increase of the pressure would cause it to reverse. It is not however the *reversibility* (in the literal sense of this word) which is the essential feature of reversible processes from the point of view of thermodynamics ; it is the succession of equilibrium states which is the important characteristic of them. The curve representing the relation between the pressure and

FIG. 15·3

volume of a substance during a reversible change at constant temperature is called an **isothermal**.

There is another relation between the pressure and volume of a substance with which we are much concerned in thermodynamics, namely the relation which subsists between these variables during a reversible change, which is subject to the condition that heat is not allowed to enter or leave the substance. The curve representing such a relation is called an **adiabatic** and such a change is called an **adiabatic change**. The term ' adiabatic ' is often employed rather loosely and carelessly to mean *any* process subject to the condition that heat is prevented from entering or leaving the substance. There are many *very different* processes which might be termed ' adiabatic ' in this wider sense. For example we might subject a gas to the condition that heat is not allowed to enter or leave it and allow it to double its volume in the following different ways : (*a*) by expanding into a previously exhausted space, (*b*) by expanding reversibly. In the former process, experiment shows that its temperature is only very slightly altered, in the latter the gas is very appreciably cooled. In this treatise the term ' adiabatic ' will be used, unless the contrary is clearly indicated, for a process subject to the *two conditions*, (i) not transfer of heat, (ii) reversibility. The latter condition means that the process takes place in such a way that the substance remains practically in a state of equilibrium.

FIG. 15·31

There are other ways of representing the states of a substance graphically. We may, for instance, represent the relation between pressure and temperature under the condition of constant volume. Such curves are called *isochores*. Or we may represent the relation between volume and temperature under the condition of constant pressure and we have the curves known as *isopiestics*. A very important example of a pressure-temperature diagram is that representing the equilibrium between different *phases* of a substance, i.e. between its solid, liquid and vapour states, or between the *phases* of a system with more than one constituent, e.g. water and common salt. The phases in this case would include ice, water vapour, the solution of the salt in water, and so on. The equilibrium between the different phases of water is illustrated in Fig. 15·31.
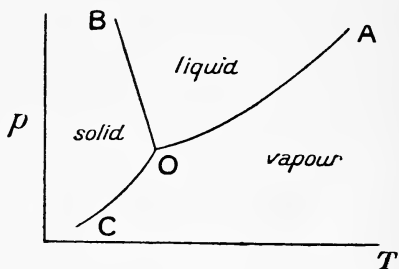
When the substance is in a state represented by any point on the line (OA), the liquid and its vapour are in equilibrium, i.e. neither evaporation nor condensation goes on. For such states both phases may exist simultaneously. If however the pressure, at some given temperature, is raised above the value corresponding to a point on (OA), the equilibrium state will be one in which only the liquid phase can exist ; if the pressure is less than the value corresponding to a point on (OA), then only the vapour phase will be possible. Similar remarks apply to the curves (OB) and (OC). The point, O, represents a pressure and temperature at which all three phases can co-exist.

§ 15·4. WORK DONE DURING REVERSIBLE EXPANSION

Let us imagine the substance to be contained in a cylinder (Fig. 15·4) closed by a piston. The pressure, p, is, by definition, the force per unit area ; so that if A represents the area of the piston, pA will be the force exerted on it during a *reversible* change. During any very small expansion the pressure and therefore the force, pA, exerted on the piston will be sensibly constant, and the work done will be equal to pAs, if s represents the distance the piston travels. The product, As, is the corresponding increase in volume, so that during a small reversible expansion (§ 12)

$$dW = pdv \qquad \ldots \ldots \quad (15·4)$$

FIG. 15·4

where dW is the work done by the substance, and dv is the corresponding small increase in volume. We see, therefore, that the work done during a *reversible* expansion from an initial volume v_1 to a final volume v_2 is expressed by the formula

$$W = \int_{v_1}^{v_2} pdv \qquad \ldots \ldots \quad (15·401)$$

This work is obviously represented on the pv diagram by the area enclosed between the perpendiculars erected at v_1 and v_2. During an isothermal expansion for instance it is represented by the shaded area in Fig. 15·3.

In the special case of the isothermal expansion of a gas, we have, since $p = \dfrac{RT'}{v}$,

$$W = RT' \int_{v_1}^{v_2} \frac{dv}{v}$$

or
$$W = RT' \log \frac{v_2}{v_1} . \quad . \quad . \quad . \quad . \quad (15·41)$$

or, since in this case
$$p_1 v_1 = p_2 v_2,$$

$$W = RT' \log \frac{p_1}{p_2} \quad . \quad . \quad . \quad (15·411)$$

The formulae are, of course, only approximately true for actual gases. If we deal with a gram-molecule of a gas and use absolute units, e.g. if we measure pressures in dynes per square centimetre and volumes in cubic centimetres, the constant R has the same value, nearly, for all gases, namely

$$R = 8·315 \times 10^7 \text{ ergs per } °C.,$$

so that the work of expansion in such a case is given by

$$W = 8·315 \times 10^7 T' \log \frac{v_2}{v_1}.$$

If we use the **practical** unit of work, the **joule,** we have obviously to give R the value 8·315 joules per degree.[1] Finally we may sometimes find it convenient to express the work in terms of the equivalent number of gram calories, in which case R will be approximately 1·98 calories per degree.

Fig. 15·41

When a substance is made to pass reversibly through a succession of states represented by a closed curve on the pv diagram, it follows from (15·401) that the net amount of work done by the substance against the external pressure, or done on it by the external pressure, according as the closed curve is described in a clockwise or counter clockwise sense, is equal to the area within the closed curve. Suppose the substance to start from the condition represented by the point A (Fig. 15·41) and to travel along the path ACB to B. The work done *by* it is represented by the area bounded by ACB and by the perpendiculars AM and BN. If it is now caused to pass along the curve BDA to its original state A, the work done *on* it will be represented by the area bounded by the curve ADB and the perpendiculars AM and BN. Therefore the excess of work done *by* the substance over that done *on* it is represented by the area of the loop.
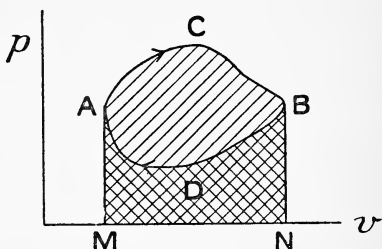
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[1] Since the joule is equal to 10^7 ergs.

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§ 15·5.   HEAT

The meanings of the terms *temperature* and *scale of temperature* have already been explained, and we have now to distinguish between the notion of temperature and that of **heat,** or **quantity of heat.**   If a piece of some metal, initially at 100° C., be dropped into a cavity in a block of ice at 0° C., thermal equilibrium will be established when the metal has cooled down to 0° C., and a definite quantity of the ice will be melted during the process. We may define heat by using the amount of ice melted to measure the quantity of heat lost by the metal.   Such a **calorimeter,** consisting of a block of ice with a cavity in it, covered by an ice lid to prevent heat from the room melting ice within the cavity, was used by Joseph Black (1728–1799) for measuring quantities of heat, and was one of the earliest, if not the earliest, forms of calorimeter.   The unit of heat, called the *calorie*, may be defined as the quantity of heat necessary to raise a gram of water 1° C. in temperature.   The calorie so defined is not a unique quantity, since experiment shows that the quantity of heat necessary to raise a gram of water from 0° C. to 1° C., for example, is *not quite* the same as that needed to raise it, say, from 20° C. to 21° C.   The term ' calorie ' is used for any of a number of units of heat, most of them differing very little from one another. The 15° calorie is the quantity of heat needed to raise a gram of water from $14\frac{1}{2}$° C. to $15\frac{1}{2}$° C. ;   the mean calorie raises 0·01 gram of water from 0° C. to 100° C. ;   the zero calorie raises a gram of water from 0° C. to 1° C. and so on.   All these units differ only slightly from one another.

§ 15·6.   FIRST LAW OF THERMODYNAMICS

It has already been pointed out that Carnot himself arrived at the great generalization known as the Principle of Conservation of Energy.   The following passage was found after his death, in 1832, among his unpublished manuscripts : *La chaleur n'est autre chose que la puissance motrice [ou plutôt que le mouvement] qui a changé de forme.   [C'est un mouvement dans les particules du corps.]   Partout où il y a destruction de puissance motrice, il y a, en même temps, production de chaleur en quantité précisément proportionelle à la quantité de puissance motrice détruite.   Reciproquement, où il y a destruction de la chaleur, il y a production de puissance motrice.*

Ou peut donc poser en thèse générale que la puissance motrice est en quantité invariable dans la nature, qu'elle n'est jamais, à proprement parler, ni produite, ni détruite.

This is a clear statement of the energy principle and Carnot's *puissance motrice* is simply what we now-a-days call **energy**. It is true that, since the advent of the theory of relativity, we have come to regard energy as something having a more ' substantial' character than the mere capacity for doing work, or *puissance motrice* ; but we are not at present concerned with this.

The general adoption of the principle of energy came about in consequence of the experimental work of J. P. Joule, a Manchester brewer,[1] who carried out a series of classical experiments between 1840 and 1850. He determined, in various ways, the amount of work which must be done to generate a unit of heat and his results differ only slightly from the best modern measurements, which yield the mean result that one 15° calorie is equivalent to $4·188 \times 10^7$ ergs. The work of Joule received important confirmation a little later by G. A. Hirn, an engineer of Colmar in Alsace, who, among other researches of interest and importance, carried out experiments on a steam engine of a converse type to those of Joule. That is to say he measured the heat used up to do work and his results showed that the mechanical equivalent is just the same as when work is done to generate heat.

The principle of conservation of energy viewed from the standpoint of Joule or Hirn, is the deliverance of an extensive series of careful experiments. It is therefore a physical law which (like that of Boyle for example) might conceivably, when the accuracy of temperature measuring devices is sufficiently improved, turn out to be an approximation only. The experiments can scarcely assure us of its *exact* validity. Nevertheless we have gradually, and perhaps uncritically, developed a belief in its perfect exactitude. Indeed if future experiments should reveal that in certain circumstances more heat is generated, for example, than the work done would require, we should hardly doubt the principle of conservation, but rather infer from such experiences a previously unsuspected source of energy.

If dQ represent a small quantity of heat communicated to a system and dW the excess of the work done by the system

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[1] The untenability of the old *caloric* theory was demonstrated before the close of the eighteenth century by Count Rumford's famous experiments on the boring of cannon at Munich, and by Sir Humphry Davy's experiments in which heat was generated by friction between blocks of ice. The former indeed furnished a rough estimate of the mechanical equivalent of heat.

RUMFORD : 'An Enquiry concerning the source of the heat which is excited by friction.' *Trans. Roy. Soc.*, Jan. 25th, 1798.
DAVY : Collected works.

over that done on it, then we have for the gain in energy of the system

$$dU = dQ - dW \quad . \quad . \quad . \quad . \quad (15\cdot6)$$

The letter U represents what is called the **internal** or **intrinsic energy** of the system. We are concerned, for the present, with systems, the equations of state of which are relations between pressure, volume and temperature ; that is to say with systems the state of which is fixed by the values of any two of these variables ; so that the internal energy of such systems is a function of the pressure and volume, or of the temperature and volume or of the pressure and temperature.

§ 15·7. Internal Energy of a Gas

Experiments carried out by Gay-Lussac as long ago as 1807 indicated that the internal energy of a gas is determined solely by its temperature. Very similar experiments were carried out by Joule independently and much later. He allowed air, contained in a copper vessel under a considerable pressure, to expand into a similar, previously exhausted vessel. The vessels were immersed in water, and Joule found no appreciable change in the temperature of the latter on stirring it after the expansion ; though he observed very marked temperature changes when the vessels were immersed in water in *separate* containers, the water surrounding the vessel out of which the air was expanding being cooled, and that surrounding the other vessel being heated. It is easy to see that the interpretation of these experiments is that given above. For no heat is communicated to or abstracted from the air during the experiments and no external work is done. Therefore by (15·6), the change in the internal energy is zero ; and since the temperature of the air as a whole is not affected we see that the internal energy is the same for different volumes at the same temperature.

A more sensitive method of investigating the dependence of the internal energy of a gas on its volume was suggested by Lord Kelvin, and carried out by him in collaboration with Joule. The results and the theory of their experiments will be dealt with in some detail later ; it will suffice to state here that the internal energy of an actual gas does vary slightly with its volume.

§ 15·8. Specific Heat

If, when a small quantity of heat dQ is communicated to a gram of a substance, there is a rise in temperature dt, we define

$$c = \frac{dQ}{dt}$$

to be the **specific heat** of the substance. It is clear that this ratio will depend on the conditions under which the heat is communicated, since we can alter the temperature of the substance quite appreciably without communicating or withdrawing heat at all ; but merely by compressing it, or allowing it to expand. We are chiefly concerned with the specific heat measured under the conditions of constant pressure (and *reversible* expansion) or of constant volume. If we use the gas scale of temperature, the specific heats of a gas are approximately constants. The specific heat of a gas at constant volume, for example,

$$c_v = \left(\frac{dQ}{dT'}\right)_v,$$

is nearly independent of the temperature and volume of the gas. This is sometimes called the **law of Clausius.**

§ 15·9. The Perfect Gas

Actual gases, we have seen, conform approximately to three laws, namely :
  i. **The law of Boyle,**
  ii. **The law of Joule, which may be expressed in the form**

$$\left(\frac{\partial U}{\partial v}\right)_{T'} = 0 \quad . \quad . \quad . \quad . \quad . \quad \textbf{(15·9)}$$

  iii. **The law of Clausius.**
We shall use the term **perfect gas** or **ideal gas** for a hypothetical gas which obeys these laws exactly.

We shall now apply the first law to a perfect gas. For a reversible process equation **(15·6)** becomes

$$dQ = dU + pdv \quad . \quad . \quad . \quad . \quad \textbf{(15·91)}$$

since the work done, dW, is now expressed by pdv. It must be remembered that in equations **(15·6)** and **(15·91)** the heat supplied, the internal energy and the work done are all expressed in terms of the same unit—which may for example be the erg. We shall often have occasion to make use of the formula

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad . \quad . \quad . \quad . \quad \textbf{(15·92)}$$

where z is a function of the independent variables x and y, and where the round ∂'s are used to indicate *partial* differentiation. In obtaining the coefficient $\frac{\partial z}{\partial x}$, for example, the other independent variable, y, is kept constant during the differentiation. Since

the internal energy, U, of a system, is a function of the temperature and volume we have, by (**15·92**)

$$dU = \left(\frac{\partial U}{\partial T'}\right)_v dT' + \left(\frac{\partial U}{\partial v}\right)_{T'} dv \quad . \quad . \quad . \quad \textbf{(15·93)}$$

where the suffixes are used to indicate the variable which is kept constant during the differentiation. Equation (**15·91**) now becomes

$$dQ = \left(\frac{\partial U}{\partial T'}\right)_v dT' + \left\{\left(\frac{\partial U}{\partial v}\right)_T + p\right\} dv \quad . \quad . \quad \textbf{(15·94)}$$

This formula is quite general. It applies to a reversible expansion of any substance. Applied to a perfect gas it takes the special form

$$dQ = \frac{\partial U}{\partial T'} dT' + p\,dv \quad . \quad . \quad . \quad \textbf{(15·941)}$$

in consequence of the law of Joule (**15·9**).

If we are dealing with a gram of the gas, (**15·941**) obviously becomes

$$dQ = c_v dT' + p\,dv \quad . \quad . \quad . \quad \textbf{(15·942)}$$

and if the heat dQ is communicated under the condition of constant pressure,

$$p\,dv = R\,dT'$$

and therefore $\quad dQ_p = c_v dT_p' + R\,dT_p'$

or $\qquad\qquad \left(\dfrac{dQ}{dT}\right)_{const.\ pressure} = c_v + R$

or $\qquad\qquad \boxed{c_p = c_v + R} \quad . \quad . \quad . \quad . \quad . \quad \textbf{(15·95)}$

Expressed in words, this formula states that the excess of the specific heat of a gas at constant pressure over that at constant volume is equal to the gas constant for a gram of the gas. If the specific heats are expressed in calories per gram per degree the formula becomes

$$c = c_v + \frac{R}{J} \quad . \quad . \quad . \quad . \quad \textbf{(15·951)}$$

where J is the number of ergs equivalent to one calorie, i.e. the mechanical equivalent of heat. This formula in fact furnishes us with a means of determining J. If we take one gram of air (which approximates very closely to a perfect gas) we have approximately

$$R = 2·9 \times 10^6 \text{ ergs per degree,}$$
$$c_p = ·239 \text{ cal. per gram per degree,}$$
and $\qquad c_v = ·169 \quad ,, \quad ,, \quad ,, \quad ,, \quad ,,$

from which we get, by substituting in (**15·951**),

$$J = 4·14 \times 10^7 \text{ ergs per cal.}$$

This is the method of determining J which was employed by Mayer in 1842, and still earlier by Carnot.

Equation (**15·942**), which governs *any* reversible change in a perfect gas, will, when applied to an adiabatic change, take the form

$$0 = c_v dT' + p dv$$

or, since we have under all circumstances,

$$p = \frac{RT'}{v},$$

$$0 = c_v dT' + RT' \frac{dv}{v}.$$

If we divide both sides of this equation by $c_v T'$ and make use of equation (**15·95**), we get,

$$0 = \frac{dT'}{T'} + (\gamma - 1)\frac{dv}{v},$$

where γ is employed for the ratio, c_p/c_v, of the specific heats at constant pressure and constant volume. When we integrate this equation we obtain

$$0 = \log \frac{T'}{T_0'} + (\gamma - 1) \log \frac{v}{v_0},$$

where T_0' and v_0 represent the initial temperature and volume and T' and v the final temperature and volume. This result may obviously be written in the form,

$$\log T' + (\gamma - 1) \log v = \log T_0' + (\gamma - 1) \log v_0,$$

or in the equivalent forms

$$\log T' + (\gamma - 1) \log v = \text{constant} \quad . \quad . \quad . \quad (15·96)$$
$$T'v^{\gamma-1} = \text{constant} \quad . \quad . \quad (15·961)$$
$$pv^\gamma = \text{constant} \quad . \quad . \quad . \quad (15·97)$$
$$T'^\gamma p^{1-\gamma} = \text{constant} \quad . \quad . \quad . \quad (15·98)$$

the two latter equations being obtained by eliminating T' and v respectively in (**15·961**) by the substitutions $T' = pv/R$ and $v = RT'/p$.

The constant γ, as defined above, is the ratio c_p/c_v. Reference to equations (**10·11**) and (**10·12**) shows that it is also equal to the ratio, e_ϕ/e_T, of the adiabatic elasticity of the gas to its isothermal elasticity. This equality is the basis of the method of Clement and Désormes for determining the ratio of the specific heats of a gas and of the method of obtaining it from the measured

velocity of sound in the gas.   By (**10·21**) the velocity of sound waves in a gas is

$$u = \sqrt{\frac{k}{\rho}}\ ,$$

since n is zero ; and the compressions and rarefactions in sound waves of audible frequency in gases are practically adiabatic, so that

$$k = \gamma p.$$

Hence

$$u = \sqrt{\gamma \frac{p}{\rho}}\ ,$$

or

$$u = \sqrt{\gamma R T}\ \big|\ .$$

§ 16.   HEAT SUPPLIED TO A GAS DURING REVERSIBLE EXPANSION

We have seen that when we subject a gas to the condition of constant temperature, the relation between its pressure and volume is expressed by

$$pv = \text{constant.}$$

If it is subjected to adiabatic conditions, the relation is expressed by

$$pv^{\gamma} = \text{constant.}$$

More generally any condition to which the behaviour of the gas is subjected will make its pressure some function of its volume,

$$p = f(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad \textbf{(16)}$$

We can deduce an expression for the heat supplied to the gas during a reversible expansion under the condition expressed by (**16**).   From the equation of state of the gas we have

$$dT' = \frac{pdv + vdp}{R},$$

and when we substitute this expression for dT' in equation (**15·942**) we get,

$$dQ = c_v \frac{pdv + vdp}{R} + pdv,$$

or

$$dQ = \frac{v}{\gamma - 1}dp + \frac{\gamma}{\gamma - 1}pdv.$$

We now eliminate dp from this last equation by means of (**16**). We have

$$dp = \frac{df(v)}{dv}dv,$$

or

$$dp = f'(v).dv,$$

and therefore

$$dQ = \frac{vf'(v)}{\gamma - 1}dv + \frac{\gamma}{\gamma - 1}pdv.$$

In the special case where

$$f(v) = \frac{c}{v^s}, \text{ or } pv^s = c,$$

c and s being constants, we have

$$f'(v) = \frac{-sc}{v^{s+1}} = -\frac{sp}{v},$$

and therefore

$$dQ = \frac{\gamma - s}{\gamma - 1}pdv \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \textbf{(16·01)}$$

or

$$(\text{Heat supplied}) = \frac{\gamma - s}{\gamma - 1} \times (\text{work done}) \quad \cdot \quad \cdot \quad \cdot \quad \textbf{(16·02)}$$

When the expansion is isothermal,

$$s = 1,$$

and we see that the heat supplied is equal to the work done, as indeed is otherwise evident from the fact that during an isothermal expansion the internal energy of a gas does not alter. If on the other hand we put

$$s = \gamma$$

we have a further verification of our formula, since it correctly states that in this case the heat supplied is zero.

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CHAPTER XIII

SECOND LAW OF THERMODYNAMICS

§ 16·1. The Perpetuum Mobile of the Second Kind

IN the treatise referred to in § 15, Carnot makes the statement : ' La production de la puissance motrice est donc due, dans les machines à vapeur, non à une consommation réelle du calorique, *mais à son transport d'un corps chaud à un corps froid, . . .*' The words in italics constitute the earliest expression of the second law of thermodynamics. The rest of the statement is founded on the erroneous principle of the conservation of heat or caloric, which found acceptance in Carnot's time, and we are not concerned with it. All heat engines, as Carnot noticed, in doing work, not only abstract heat from a source of heat ; but give up a portion of it to a region (condenser or surrounding atmosphere) where the temperature is lower than that of the source of heat. *In practice* it is found to be impossible to consume heat from a source in doing work, without giving up some of it to a condenser, or something, at a lower temperature. It is true that a limited amount of work can be done simply at the expense of heat taken from a source *without* giving heat to any other body, as for example during the expansion of a gas. But an expansion cannot be extended indefinitely, and actual engines are machines which necessarily work in a cyclic fashion, and during some part of the cycle heat is always rejected. Were it not for this sort of limitation of the convertibility of heat into work, the practicability of propelling ships at the expense of the heat in the surrounding sea might be contemplated. Following Planck, we shall provisionally regard the second law as equivalent to the statement :

It is impossible to construct an engine which
i. **repeats periodically a cycle of operations,**
ii. **raises a weight,**
iii. **takes heat from a source of heat and does nothing else.**

The kind of machine which this axiom declares to be an impossibility is called by Ostwald a **perpetuum mobile** of the

second kind to distinguish it from another type of impossible machine, namely one which simply does work gratis, or without the consumption of energy at all, and which may be called a **perpetuum mobile** of the **first kind.**

§ 16·2. CARNOT'S CYCLE

In order to make use of this axiom, we shall study an ideal type of heat engine first described in Carnot's treatise. It consists of a cylinder, A, (Fig. 16·2) and a piston, B, both made of material which is thermally perfectly insulating. The base, C, of the cylinder, is made of conducting material. Further, the piston can slide in the cylinder without any frictional resistance whatever. It is connected with ideal frictionless machinery, so as to enable it to raise a weight. The source of heat, X,
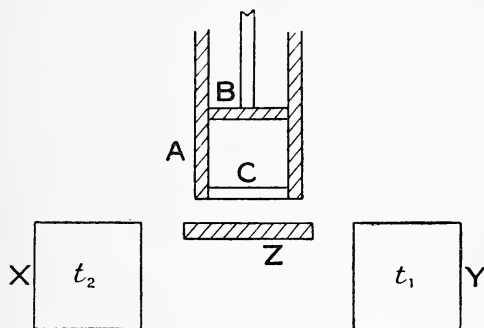
FIG. 16·2

at the temperature t_2 (expressed in terms of some arbitrary scale) is supposed to be a perfectly conducting block of material, with a practically infinite heat capacity. There is a similar block of material, Y, at a lower temperature, t_1, which we shall call the refrigerator. A block of thermally perfectly insulating material, Z, can be used at certain stages in the periodic working of the engine to cover the lower end of the cylinder. *No assumptions are made concerning the nature of the working substance*, except that it must be capable of exerting a pressure on the piston. It may be a gas, a mixture of water and its vapour, or anything else which might be used to operate an actual engine.

Let us suppose the engine to begin work with its working substance in the state represented by the point 1 on the indicator diagram (Fig. 16·21). The base of the cylinder is covered by the source of heat, X, (Fig. 16·2), and the load is so adjusted that the upthrust on the piston exceeds by an infinitesimal amount

the force necessary to balance the downward thrust due to the load. Under these circumstances the substance expands isothermally at the temperature t_2. After a suitable expansion, corresponding to the point 2 on the diagram, the source is removed and the cylinder covered by the slab Z. The working substance now expands adiabatically, its temperature being steadily reduced till it reaches the state 3 on the diagram corresponding to the temperature t_1. The slab Z is now removed and the block Y brought into contact with the base of the cylinder. An infinitesimal readjustment of the load is now made, so that the piston descends with extreme slowness. The working substance is now compressed reversibly and isothermally. This is allowed to continue till it reaches the state 4, and then the block Y is replaced by Z and the compression is continued adiabatically till the substance reaches its original state.
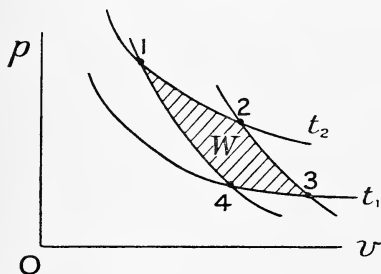
FIG. 16·21

We may define the **efficiency** of an engine as the work done during a cycle divided by the corresponding quantity of heat taken from the source. In the case of the reversible engine just described, the work done during a cycle is equal (§ 15·4) to the area, W, of the closed curve (1, 2, 3, 4) on the indicator diagram (Fig. 16·21). We have therefore

$$\text{Efficiency} = \frac{W}{Q_2}. \quad . \quad . \quad . \quad (16\cdot2)$$

where Q_2 is the heat supplied by the source at the temperature t_2. Since the working substance returns to its original state at the end of the cycle, the first law (**15·6**) requires that

$$Q_2 - Q_1 = W \quad . \quad . \quad . \quad . \quad (16\cdot201)$$

and hence

$$\text{Efficiency} = \frac{Q_2 - Q_1}{Q_2} \quad . \quad . \quad . \quad (16\cdot21)$$

where Q_1 is the heat rejected to the refrigerator at the temperature t_1.

§ 16·3. CARNOT'S PRINCIPLE

We shall now prove that the axiom of § 16·1 leads to the consequence that all reversible engines working between the same temperatures, t_2 and t_1, have the same efficiency ; or, in other words, that the efficiency of a reversible engine depends on the

temperatures of the source and the refrigerator and on nothing else. Let us suppose that, of two reversible engines A and B, working between the temperatures t_2 and t_1, A has the greater efficiency and let us provisionally suppose further that both engines take the same quantity of heat Q_2 from the same source during a cycle, and that they use the same refrigerator. We have then

$$\frac{W_A}{Q_2} > \frac{W_B}{Q_2}, \text{ by hypothesis,}$$

and therefore

$$W_A > W_B, \quad . \quad . \quad . \quad . \quad . \quad (16\cdot3)$$

where W_A and W_B represent the work done during a cycle by the engines A and B respectively. It follows from (16·201) that A rejects to the refrigerator a smaller quantity of heat during a cycle than does B. Let us now imagine the two engines to be coupled together by ideal machinery (i.e. frictionless machinery), so that A drives B backwards and makes it exactly reverse its normal operations in such a way that the two engines complete their cycles in equal times. This is possible because of the reversible character of B, and because of the inequality (16·3).

The circumstance that during certain stages of this compound cycle, work is actually being done *on* the engine A, or indeed on both engines at the same time, need cause us no difficulty. We have only to think of the ideal machinery as suitably controlled by a fly-wheel with an enormous moment of inertia. It is clear that the ' source ' at the temperature t_2 will now change in a way which is exactly periodic, the period being equal to that of either engine (say τ); since during such a period A removes Q_2 units of heat from it, while B restores the same amount to it. The ' refrigerator ' on the other hand has more heat abstracted from it by B during the period, τ, than is restored to it by A. Of the work, W_A, done by A, the portion, W_B, is used in driving B backwards, and the balance, $W_A - W_B$, may be applied to raise a weight. The combination of A and B *and the ' source '* at the temperature t_2 constitutes an engine which

    i. repeats periodically a cycle of operations,
    ii. raises a weight,
    iii. takes heat from a source of heat (in this case from what, in the normal working of A and B, has been called the ' refrigerator ') and does nothing else.

This is in conflict with the axiom of § 16·1, and therefore the hypothesis that the engine A has a greater efficiency than B is an untenable one. They must have the same efficiency.

We have restricted ourselves to the case of engines taking the same quantity Q_2 from the source during a cycle. We can however easily prove that the efficiency of a reversible engine is independent of the quantity of heat taken from the source during a cycle. Suppose we have a reversible engine working round the cycle *abcd* (Fig. 16·3) between the isothermals t_2 and t_1. Let q be the quantity of heat taken from the source at t_2, and w the work done during a cycle. Its efficiency is therefore w/q. If the engine be adjusted so as to work round the cycle *befc* between the same isothermals t_2 and t_1, as before, its efficiency will not be altered provided it still takes the same quantity of heat q from the source. It follows that the work done during a cycle is also the same as before, i.e. the two areas *abcd* and *befc* are each equal to w. Now let the engine be adjusted to work round the cycle *aefd*. Its efficiency is equal to the area of the closed loop *aefd* divided by the heat it abstracts from the source. That is to say, it is equal to $2w/2q = w/q$. So that doubling the quantity of heat it takes from the source does not affect its efficiency. A very obvious extension of this proof leads to the conclusion that if the engine is adjusted so as to modify in any way whatever the quantity of heat it removes from the source during a cycle of operations its efficiency will not be affected and **Carnot's principle** is established.
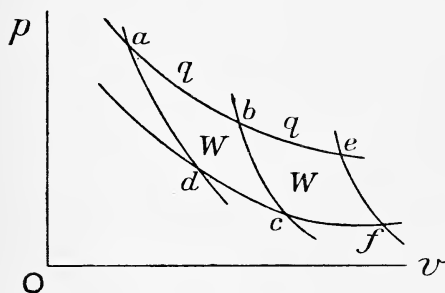
FIG. 16·3

§ 16·4. KELVIN'S WORK SCALE OF TEMPERATURE

Carnot's principle enables us to define a scale of temperature which is quite independent of the nature of any of the physical quantities, or of the apparatus used in measuring temperatures. If we consider a number of reversible engines, all of which work between the same temperatures t_2 and t_1, which we may suppose, for the present, to be measured in terms of some arbitrary scale, we have

$$\frac{Q_2 - Q_1}{Q_2} = \frac{Q_2' - Q_1'}{Q_2'} = \frac{Q_2'' - Q_1''}{Q_2''} = \text{etc.},$$

or

$$\frac{Q_2}{Q_1} = \frac{Q_2'}{Q_1'} = \frac{Q_2''}{Q_1''} = \text{etc.} \quad . \quad . \quad . \quad \textbf{(16·4)}$$

This means that if a substance in expanding isothermally, at the temperature t_2, absorbs a quantity of heat, Q_2; and in expanding isothermally at another temperature, t_1, between the same two adiabatics, absorbs the quantity of heat Q_1, the ratio $\dfrac{Q_2}{Q_1}$ is independent of the nature of the substance, and also of the pair of adiabatics chosen, and depends solely on the temperatures t_2 and t_1. In what follows we shall usually employ this result as an axiom, in place of the axiom (16·1). It may be regarded as equivalent to the second law of thermodynamics. We shall now define a scale of temperature by the equation

$$\frac{Q_2}{Q_1} = \frac{T_2}{T_1} \quad . \quad . \quad . \quad . \quad . \quad \textbf{(16·41)}$$

We can show that the scale so defined is independent of the particular substance which may absorb the quantities of heat Q_2 and Q_1, when expanding isothermally between the same pair of adiabatics. The ratio of the same two temperatures on the scales defined by different substances, using (16·41), is the same for all substances, i.e.

$$\frac{T_2}{T_1} = \frac{T_2'}{T_1'} = \frac{T_2''}{T_1''} \quad . \quad . \quad . \quad . \quad \textbf{(16·42)}$$

because of (16·4). Now we have agreed that the temperature difference between saturated steam at normal pressure and melting ice at the same pressure shall be numerically 100, therefore we get, when we apply (16·42) to these two temperatures,

$$\frac{T_0 + 100}{T_0} = \frac{T_0' + 100}{T_0'} = \frac{T_0'' + 100}{T_0''} = \text{etc.,}$$

where T_0, T_0', T_0'', etc., represent the temperature of the melting ice on the scales defined by different substances. We see that

$$T_0 = T_0' = T_0'' = \text{etc.} \ldots$$

It is clear, therefore, that the temperature of melting ice, measured on a scale defined in this way, is independent of the properties of the thermometric substance involved. We can now show very simply that this is true of any other temperature, for since

$$\frac{T}{T_0} = \frac{T'}{T_0'} = \frac{T''}{T_0''} = \text{etc.,}$$

or, using the result just obtained,

$$\frac{T}{T_0} = \frac{T'}{T_0} = \frac{T''}{T_0} = \text{etc.,}$$

therefore

$$T = T' = T'' = \text{etc.}$$

It is obvious that if we use this scale of temperature, the efficiency of a reversible engine is expressed by

$$\frac{T_2 - T_1}{T_2},$$

where T_2 and T_1 are the temperatures of source and refrigerator respectively. The scale we have just described, and which we owe to Lord Kelvin, may be described in another way. Let us imagine any pair of adiabatics, *abcd* and *efgh* (Fig. 16·4) of some substance constructed, and also the isothermals corresponding to the temperatures of steam and melting ice, which we may conveniently number 100° C. and 0° C. Now construct isothermals to divide the area *bfgc* into 100 equal parts, the area of each of which we may call ϕ. If we number them in order 1°, 2°, 3° . . . 99° C. and continue them below 0° C. and above 100° C. in the same way, that is, so as to have the same area, ϕ, between consecutive isothermals and the pair of adiabatics, we shall have the Kelvin scale of temperature, except for the trivial difference that we have numbered the temperatures from that of melting ice as a zero. This is obviously the case since (**16·41**) gives us
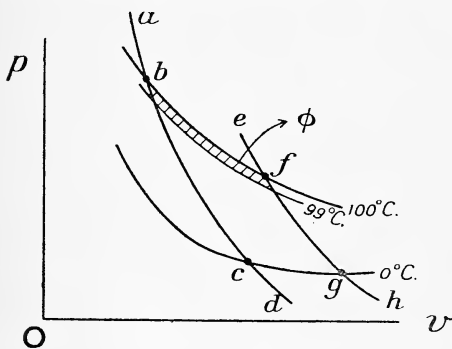
Fig. 16·4

$$Q_2 = \phi T_2$$
$$Q_1 = \phi T_1 \quad . \quad . \quad . \quad . \quad (16\cdot43)$$

where ϕ is the same constant for the same pair of adiabatics, and therefore

$$Q_2 - Q_1 = \phi(T_2 - T_1),$$

or if we apply this to the steam and ice isothermals

$$Q_{\text{steam}} - Q_{\text{ice}} = 100 \ \phi,$$

therefore

$$\phi = \frac{Q_{\text{steam}} - Q_{\text{ice}}}{100},$$

or

$$\phi = \frac{\text{area } bfgc}{100} \quad . \quad . \quad . \quad . \quad (16\cdot431)$$

Equations (**16·41**) indicate that the zero isothermal on the Kelvin scale is characterized by the property that no heat is

absorbed by the substance in passing from one adiabatic to another at this temperature. A reversible engine working round a cycle bounded by two adiabatics and the isothermals T and zero would consume all the heat absorbed at the temperature T in doing work, since none is rejected to the refrigerator at the temperature zero. Since the first law requires that more work than is equivalent to the heat supplied cannot be done in a Carnot cycle we must conclude that the zero on the Kelvin scale is the lowest of all temperatures. It is called the **absolute zero.**

The **Kelvin,** or **work scale** of temperature as it is sometimes called, is not the only absolute scale of temperature. There is an infinite number of such scales. We may for example define a scale of temperature by laying down that the efficiency of a Carnot engine, working in a cycle bounded by any two adiabatics, and by a pair of isothermals which are very close together, is proportional to the temperature difference between the isothermals.[1] This means, if we use θ to represent temperatures on this scale,

$$\frac{dQ}{Q} = ad\theta,$$

where dQ is the excess of heat absorbed over that rejected, and a is a constant. We have therefore

$$\frac{dT}{T} = ad\theta,$$

or
$$T = Ce^{a\theta},$$

where C is a constant of integration. We may choose such a value for the constant a as will make the temperature difference between melting ice and steam 100, and for the constant, C, a value which will make *one* temperature, say that of melting ice, the same on both scales. If we do this, a is given by

$$\frac{T_{\text{steam}}}{T_{\text{ice}}} = e^{100a},$$

or
$$a = \frac{1}{100} \log \frac{T_{\text{steam}}}{T_{\text{ice}}};$$

and C is fixed by

$$T_{\text{ice}} = Ce^{aT_{\text{ice}}}$$

or
$$C = \frac{T_{\text{ice}}}{e^{aT_{\text{ice}}}}.$$

We see that the temperature corresponding to the Kelvin abso-

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[1] This scale was in fact proposed by Kelvin before the work scale.

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lute zero is represented by minus infinity on the new scale. There is a certain appropriateness about this ; since the so-called absolute zero is very difficult to approach, and indeed there is reason to suspect that it is a temperature which is unattainable.

§ 16·5.   The Work Scale and the Gas Scale

The real merit of the work scale, and the reason for preferring it to any other of the possible alternatives, lie in the fact that it is identical with the perfect gas scale, and therefore approximates very closely to the temperatures as given by a gas thermometer containing hydrogen or some gas differing little from a perfect gas. The temperatures as given by such a thermometer therefore require only very small corrections to convert them to the work scale. We can prove this in the following way : If Q_2 and Q_1 represent the quantities of heat absorbed by a substance in expanding isothermally and reversibly from one adiabatic to another at the temperatures T_2 and T_1 respectively, that is say if Q_2 represents the heat absorbed by a substance expands from the point 1 to the point 2 (Fig. 16·21), and Q_1 that absorbed during an expansion from the point 4 to the point 3, then, as we have seen,

$$\frac{Q_2}{Q_1} = \frac{T_2}{T_1}.$$

This is true for *any* substance and therefore true for a perfect gas. In the case of a perfect gas, however,

$$\frac{Q_2}{Q_1} = \frac{RT_2' \log \dfrac{v_2}{v_1}}{RT_1' \log \dfrac{v_3}{v_4}} \quad . \quad . \quad . \quad . \quad (16 \cdot 5)$$

by (15·41), since the internal energy does not change. Here T_2' and T_1' represent *on the gas scale* the same temperatures as T_2 and T_1 respectively. If we apply (15·96) to the adiabatic passing through 1 and 4 (Fig. 16·21), we have

$$\log T_2' + (\gamma - 1) \log v_1 = \log T_1' + (\gamma - 1) \log v_4,$$

and by applying it to the adiabatic through 2 and 3, we have

$$\log T_2' + (\gamma - 1) \log v_2 = \log T_1' + (\gamma - 1) \log v_3.$$

Subtracting the first of these equations from the second, we get

$$\log \frac{v_2}{v_1} = \log \frac{v_3}{v_4},$$

and therefore equation (16·5) becomes

$$\frac{Q_2}{Q_1} = \frac{T_2'}{T_1''},$$

hence

$$\frac{T_2}{T_1} = \frac{T_2'}{T_1''},$$

which means that the two scales of temperature are identical.

§ 16·6.   ENTROPY

We shall now introduce a quantity to distinguish the adiabatics—the term is used in the restricted sense explained in § 15·3—on the p, v diagram, just as temperature distinguishes the isothermals. This quantity is called **entropy,** a term introduced by Clausius (see § 13·4), to whom the conception of entropy is due. We may assign the value zero to the entropy of an arbitrarily chosen adiabatic, e.g., the adiabatic through the point $p_0 v_0$, where p_0 is the normal pressure, and v_0 the volume of the substance at normal pressure and temperature ; just as on the Centigrade scale we assign the value zero to the temperature of the isothermal through the same point. Having adopted an **entropy scale,** ϕ, it becomes obvious that the state of a
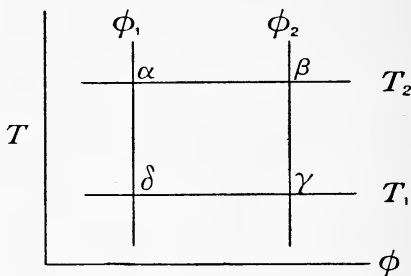
FIG. 16·6

substance (or system) in equilibrium will be determined by the corresponding values of T and ϕ, since each pair of values T, ϕ is associated uniquely with a corresponding point p, v on the p, v diagram ; and it will be helpful sometimes to employ a T, ϕ diagram instead of a p, v or other diagram. The most convenient scale for ϕ is that already defined by (16·43) or (16·431). If in Fig. 16·4, *abcd* is the adiabatic of zero entropy, the area defined by (16·431), with the $+$ or $-$ sign, according as the corresponding Q is positive or negative, will be the entropy of the substance when it is in any of the states represented by points on the adiabatic *efgh*. Or more generally the difference, $\phi_2 - \phi_1$, of the entropies associated with two adiabatics is equal to the area on the p, v diagram enclosed between the adiabatics and any pair of isothermals, the corresponding temperatures of which, on the work scale, differ by unity. Consequently

$$(\phi_2 - \phi_1)(T_2 - T_1) \quad . \quad . \quad . \quad . \quad (16·6)$$

is equal to the area of the closed curve on the indicator diagram (e.g. 1, 2, 3, 4 in Fig. 16·21) of a Carnot cycle, between the temperatures T_1 and T_2 and the adiabatics ϕ_1 and ϕ_2. So that the rectangular area, $\alpha\beta\gamma\delta$, on the T, ϕ diagram (Fig. 16·6) is equal to the corresponding area on the p, v, or indicator diagram; and it follows that the area of any closed curve on the p, v diagram is equal to the area of the corresponding curve on the T, ϕ diagram; since the former can be regarded as built up of infinitesimal elements formed by an infinite number of isothermals and adiabatics, while the latter can be regarded as built up of corresponding infinitesimal rectangles.

§ 16·7. Entropy and the Second Law of Thermodynamics

According to the definition of entropy which we have adopted (**16·43**)

$$\phi_2 - \phi_1 = \frac{Q}{T} \quad . \quad . \quad . \quad . \quad (\mathbf{16 \cdot 7})$$

where ϕ_1 and ϕ_2 are the entropies of a substance in two different equilibrium states 1 and 2. Q is the quantity of heat, positive or negative, that must be supplied to the substance *in a reversible way along any isothermal whatever* from a point on the adiabatic through 1 to the corresponding point on the adiabatic through 2, and T is the temperature of this isothermal on Kelvin's work scale. The possibility of expressing the entropy difference between two adiabatics in this way (**16·7**) is clearly a consequence of the second law and the adoption of Kelvin's work scale. Conversely we may deduce the second law (as expressed in § 16·4) from the statement (**16·7**). For consider any pair of adiabatics with the entropies ϕ_1 and ϕ_2 ($\phi_2 > \phi_1$). Then

$$\phi_2 - \phi_1 = \frac{Q_2}{T_2} = \frac{Q_1}{T_1}, \text{ by (16·7)},$$

where Q_2 is the heat communicated to the system during a reversible isothermal change from the adiabatic 1 to the adiabatic 2 at the temperature T_2, and Q_1 has a corresponding meaning for such an isothermal change at the temperature T_1. Now consider any other pair of adiabatics, of the same or any other system, with entropies ϕ_1' and ϕ_2'. We have

$$\phi_2' - \phi_1' = \frac{Q_2'}{T_2} = \frac{Q_1'}{T_1},$$

in which the significance of Q_2' and Q_1' is obvious. It follows that

$$\frac{Q_2}{Q_1} = \frac{Q_2'}{Q_1'} = \frac{T_2}{T_1}.$$

But this is the statement of the second law of thermodynamics as given in § 16·4. Consequently (**16·7**) is equivalent to the second law.

Let A and B be two neighbouring points on the p, v diagram, and let AC and BC be an isothermal through A, and an adiabatic through B respectively; their point of intersection being C (Fig. 16·7). The net amount of heat communicated to the substance during the reversible cycle ABCA is equal to the area ABC, i.e.

$$\text{Area } ABC = dQ_{AB} + dQ_{BC} + dQ_{CA},$$

or $\text{Area } ABC = dQ_{AB} + dQ_{CA},$

since BC is an adiabatic. In the limit when B and C approach
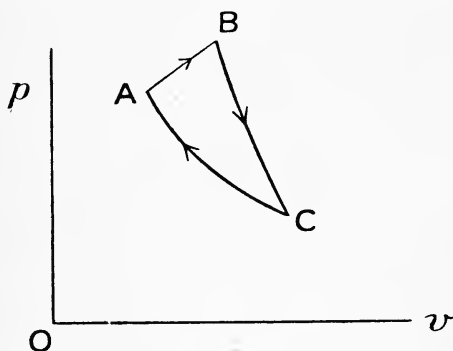
FIG. 16·7

very near to A, the area ABC becomes vanishingly small by comparison with dQ_{AB} or dQ_{CA}, since it ultimately diminishes in the same way $(AB)^2$ or $(AC)^2$; whereas dQ_{AB} or dQ_{CA} diminish as AB or AC. Therefore

$$dQ_{AB} + dQ_{CA} = 0,$$

or $dQ_{AB} = dQ_{AC},$

in the limit. Dividing both sides by the temperature, T, corresponding to the isothermal through A, we get

$$\frac{dQ_{AB}}{T} = \frac{dQ_{AC}}{T}.$$

The right-hand side of this equation represents, according to (**16·7**), the increase in entropy when the substance changes (reversibly) from the state A to the state B. We may therefore write

$$d\phi_{AB} = \frac{dQ_{AB}}{T},$$

or, simply $d\phi = \dfrac{dQ}{T}$ . . . . . (**16·71**)

Consequently the increase in entropy of a substance, or system, in changing reversibly from a state 1 to another state 2 is expressed by

$$\phi_2 - \phi_1 = \int_1^2 \frac{dQ}{T} \quad . \quad . \quad . \quad (16\cdot711)$$

and the value of the integral is clearly independent of the path joining the points 1 and 2 on the p, v diagram. An alternative expression for $\phi_2 - \phi_1$ is

$$\phi_2 - \phi_1 = \int_1^2 \frac{dU + pdV}{T} \quad . \quad . \quad . \quad (16\cdot72)$$

which, for the special case of constant volume, reduces to

$$\phi_2 - \phi_1 = \int_1^2 \frac{dU}{T} \quad . \quad . \quad . \quad (16\cdot721)$$

We shall adopt (16·71) as a final statement of the second law.[1]

§ 16·75. ENTROPY OF A GAS

For the unit mass of a perfect gas we have

$$\frac{dQ}{T} = c_v \frac{dT}{T} + R\frac{dV}{V}$$

or $\qquad d\phi = c_v d \log T + R d \log V$

and therefore $\quad \phi = c_v \log \dfrac{T}{T_0} + (c_p - c_v) \log \dfrac{V}{V_0} \quad . \quad . \quad (16\cdot75)$

if we agree that ϕ shall be zero when the temperature and volume are T_0 and V_0 respectively.

§ 16·8. PROPERTIES OF THE ENTROPY FUNCTION. THERMODYNAMICS AND STATISTICAL MECHANICS.

It is well to bear in mind that the systems with which we are dealing are characterized by an equation of state which expresses a distinctive variable, the temperature, as a function of the pressure and volume, when the system is in equilibrium. There are also systems in which there are other variables y_1x_1, $y_2x_2, \ldots y_sx_s, \ldots$ besides (or instead of) p and V. The

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[1] As we have seen, the dQ in (16·71) and (16·711) is not *any* dQ, but the special increment associated with a *reversible* process. No such cautionary remark is necessary about (16·72) or (16·721) because $dU + pdV$ represents just this particular increment dQ that is in question.

external work done during a reversible change in such a system is expressed by $\int p\,dV$ or $\Sigma \int y_s\,dx_s$. It will be convenient to call such systems **thermodynamic systems,** and we shall use the term **closed system** for one which does not interact in any way with *thermodynamic* systems outside it. A reversible process in a thermodynamic system is merely a **limit** that **actual** processes may approach—sometimes quite closely—but these latter are essentially irreversible. Mere transfer of heat—*apart from volume changes*—simply increases the internal energy of one part, a, of a system at the expense of that of another, β; the consequent (algebraic) increment of entropy being, according to (**16·721**),

$$\frac{dU_a}{T_a} + \frac{dU_\beta}{T_\beta},$$

or since

$$dU_a = -\,dU_\beta,$$

the increment of entropy is

$$\frac{dU_a}{T_a} - \frac{dU_a}{T_\beta}.$$

This is necessarily a *positive* quantity since, if dU_a is positive, T_β must be greater than T_a and, if dU_a is negative, T_a must be greater than T_β. If an irreversible process in a closed system is associated with a change in volume, the internal energy of the system is bound to be greater when the final volume is reached than it would have been had the change occurred reversibly. If it were an expansion, for instance, the resisting pressure would be less at each stage of the process than would be the case during reversible expansion. Less external work is done therefore in a given irreversible increase in volume than when the same expansion occurs reversibly, with the consequence that in the former case the final value of the internal energy is greater. Similarly during an irreversible diminution in volume the external pressure is greater at each stage than that operating when the same diminution in volume is brought about reversibly, and again the final value of the internal energy is greater in the case of the irreversible process. Let U_0 be the final value of the internal energy when the given increase in volume occurs reversibly and U its value when it occurs irreversibly, then

$$U > U_0$$

and

$$\int_{U_0}^{U} \frac{dU}{T}$$

is necessarily positive. But this integral, according to (16·721) represents the amount by which the entropy at the end of the irreversible process exceeds that at the end of the reversible process. In the latter process there is no change in entropy, consequently **the irreversible process is necessarily accompanied by an increase in the entropy of the system.** This result is quite general. In the words of Clausius:

> Die Energie der Welt ist constant.
> Die Entropy der Welt strebt einem Maximum zu.

It follows from the foregoing discussion that the necessary and sufficient condition for the equilibrium of a closed thermodynamic system is: when some small change, δ—for example a small change δV due to a slight readjustment of the external pressure—is made in the state of the system,

$$\delta\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad (16\cdot8)$$

where ϕ is the total entropy of the system. The condition is necessary because reversible changes, which as we have seen consist of successive equilibrium states, are characterized by $\phi = \text{constant}$, and it is sufficient, because no departure from equilibrium is possible unless

$$\delta\phi > 0.$$

We have now brought to light the essential identity of the entropy, ϕ, of a thermodynamical system and the function represented by ψ in Chapter XI; and a brief comparison of statistical mechanics and thermodynamics will not be out of place here. Thermodynamics rests on two main principles, which we may conveniently call the principles of energy and of entropy. It is characteristic of its methods that no hypotheses concerning the nature of heat or the microscopic or sub-microscopic constitution of materials or systems are employed. Thermodynamics therefore enables us to arrive at reliable conclusions—reliable because of the proved reliability of the two main principles—which are quite independent of the (sub-microscopic) constitution of materials and of the nature of the processes occurring in them. Statistical mechanics accomplishes something more than this. It starts out from the hypothesis that the special form of energy called heat is identical with mechanical energy [1] and bases the first law of thermodynamics on the mechanical principle of conservation of energy; while the second law of thermodynamics and the entropy function emerge as statistical

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[1] This does not necessarily mean 'mechanical' in the restricted Newtonian or Hamiltonian sense.

properties of assemblages of vast numbers of mechanical systems which interact on one another in a random fashion.

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CHAPTER XIV

THE APPLICATION OF THERMODYNAMICAL PRINCIPLES

§ 16·9. General Formulae for Homogeneous Systems

WHEN a substance has an equation of state which is a relation between T, p and V, we have seen that its entropy, ϕ, is a quantity which is uniquely determined by any two of these variables, i.e.

$$\phi = \text{function } (T, V), \text{ or } \phi = \text{function } (p, V),$$

and it follows, if we write

$$d\phi = AdT + BdV \quad . \quad . \quad . \quad . \quad (16\cdot9)$$

that

$$A = \left(\frac{\partial \phi}{\partial T}\right)_V,$$

$$B = \left(\frac{\partial \phi}{\partial V}\right)_T \quad . \quad . \quad . \quad . \quad (16\cdot901)$$

(see the formula (15·92)). Now, as we have seen, we may also write dQ, the quantity of heat communicated reversibly to the substance in a similar way :

$$dQ = A'dT + B'dV \; ;$$

but we *may not* in this case infer

$$A' = \left(\frac{\partial Q}{\partial T}\right)_V,$$

$$B' = \left(\frac{\partial Q}{\partial V}\right)_T.$$

These equations would imply that Q is a function of T and V. We have seen however that this is not the case. In fact, Q, the algebraic sum of the quantities of heat that may have been communicated to a substance, may have *any value whatever* while the independent variables that determine its state remain unchanged. We have only to recall the fact that, after completing any Carnot cycle, the variables T and V, for example, re-

assume their original values, while Q may have increased by a perfectly arbitrary quantity determined by the dimensions of the cycle (§ 16·2).

Such a differential as $d\phi$ is called a **perfect**, or **complete differential.** From (**16·901**) we derive the equation

$$\left(\frac{\partial A}{\partial V}\right)_T = \left(\frac{\partial B}{\partial T}\right)_V \quad . \quad . \quad . \quad . \quad \textbf{(16·91)}$$

which will serve us as a useful rule, when we meet with expressions like (**16·9**), which are complete differentials. Writing (**15·94**) in the form

$$dQ = \left(\frac{\partial U}{\partial T}\right)_V dT + \left\{\left(\frac{\partial U}{\partial V}\right)_T + p\right\}dV \quad . \quad . \quad \textbf{(16·92)}$$

we have $d\phi = \dfrac{1}{T}\left(\dfrac{\partial U}{\partial T}\right)_V dT + \dfrac{1}{T}\left\{\left(\dfrac{\partial U}{\partial V}\right)_T + p\right\}dV.$

Therefore, by (**16·91**),

$$\frac{1}{T}\frac{\partial^2 U}{\partial V \partial T} = -\frac{1}{T^2}\left\{\left(\frac{\partial U}{\partial V}\right)_T + p\right\} + \frac{1}{T}\frac{\partial^2 U}{\partial T \partial V} + \frac{1}{T}\left(\frac{\partial p}{\partial T}\right)_V$$

or $\left\{\left(\dfrac{\partial U}{\partial V}\right)_T + p\right\} = T\left(\dfrac{\partial p}{\partial T}\right)_V \quad . \quad . \quad . \quad \textbf{(16·93)}$

Substituting this result in (**16·92**) we have

$$dQ = \left(\frac{\partial U}{\partial T}\right)_V dT + T\left(\frac{\partial p}{\partial T}\right)_V dV,$$

or, if we are dealing with the unit mass of the substance,

$$dQ = c_v dT + T\left(\frac{\partial p}{\partial T}\right)_V dV \quad . \quad . \quad . \quad . \quad \textbf{(16·94)}$$

In this equation it is of course understood that dQ is communicated reversibly. If we further subject it to some condition, x, which might, for example, be constant volume, or constant pressure, and divide both sides by dT, we get

$$\left(\frac{dQ}{dT}\right)_x = c_v + T\left(\frac{\partial p}{\partial T}\right)_v \left(\frac{\partial V}{\partial T}\right)_x,$$

or $c_x = c_v + T\left(\dfrac{\partial p}{\partial T}\right)_v \left(\dfrac{\partial V}{\partial T}\right)_x \quad . \quad . \quad . \quad \textbf{(16·95)}$

where c_x means the specific heat of the substance measured under the condition x.

Let us now apply the same method when the independent variables are T and p. We find, since dU and dV are perfect differentials,

$$dQ = \left(\frac{\partial U}{\partial T}\right)_p dT + \left(\frac{\partial U}{\partial p}\right)_T dp + p\left\{\left(\frac{\partial V}{\partial T}\right)_p dT + \left(\frac{\partial V}{\partial p}\right)_T dp\right\},$$

or

$$dQ = \left\{ \left(\frac{\partial U}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p \right\} dT + \left\{ \left(\frac{\partial U}{\partial p}\right)_T + p\left(\frac{\partial V}{\partial p}\right)_T \right\} dp \quad \text{(16·96)}$$

Assuming the unit mass of the material, this may obviously be written

$$dQ = c_p dT + \left\{ \left(\frac{\partial U}{\partial p}\right)_T + p\left(\frac{\partial V}{\partial p}\right)_T \right\} dp \quad \text{(16·961)}$$

For $d\phi$ we have

$$d\phi = \frac{1}{T}\left\{ \left(\frac{\partial U}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p \right\} dT + \frac{1}{T}\left\{ \left(\frac{\partial U}{\partial p}\right)_T \right.$$
$$\left. + p\left(\frac{\partial V}{\partial p}\right)_T \right\} dp \quad . \quad \text{(16·962)}$$

On applying the rule of (16·91), we easily get

$$\left\{ \left(\frac{\partial U}{\partial p}\right)_T + p\left(\frac{\partial V}{\partial p}\right)_T \right\} = - T\left(\frac{\partial V}{\partial T}\right)_p \quad . \quad . \quad \text{(16·97)}$$

and hence, by (16·961),

$$dQ = c_p dT - T\left(\frac{\partial V}{\partial T}\right)_p dp \quad . \quad . \quad . \quad \text{(16·971)}$$

If again we suppose the reversible communication of heat dQ to be subject to some condition, x, and divide both sides by dT we get

$$c_x = c_p - T\left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial p}{\partial T}\right)_x \quad . \quad . \quad \text{(16·972)}$$

When the condition, x, is that of constant pressure, the formula (16·95) leads to

$$c_p - c_v = T\left(\frac{\partial p}{\partial T}\right)_v \left(\frac{\partial V}{\partial T}\right)_p \quad . \quad . \quad . \quad \text{(16·98)}$$

and we arrive at precisely the same result from (16·972), when x means constant volume.

We can readily verify that this result is in agreement with (15·95), which applies to a perfect gas ; for in this case the equation of state is

$$pV = RT,$$

and consequently

$$p\left(\frac{\partial V}{\partial T}\right)_p = R,$$

$$V\left(\frac{\partial p}{\partial T}\right)_v = R ;$$

whence it follows that (16·98) becomes

$$c_p - c_v = \frac{TR^2}{pV} = R.$$

We may express $c_p - c_v$ in terms of such quantities as the coefficient of expansion of the substance and its isothermal elasticity, which are more immediate results of experimental measurement than are the quantities $\left(\dfrac{\partial p}{\partial T}\right)_v$ or $\left(\dfrac{\partial V}{\partial T}\right)_p$. By definition the coefficient of expansion, a, is

$$a = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p,$$

and therefore

$$aV = \left(\frac{\partial V}{\partial T}\right)_p \quad . \quad . \quad . \quad . \quad . \quad (16\cdot981)$$

We may get rid of $\left(\dfrac{\partial p}{\partial T}\right)_v$ in the formula by the following device :

$$dV = \left(\frac{\partial V}{\partial T}\right)_p dT + \left(\frac{\partial V}{\partial p}\right)_T dp,$$

therefore

$$0 = \left(\frac{\partial V}{\partial T}\right)_p + \left(\frac{\partial V}{\partial p}\right)_T\left(\frac{\partial p}{\partial T}\right)_v.$$

Consequently

$$\left(\frac{\partial p}{\partial T}\right)_v = -\left(\frac{\partial V}{\partial T}\right)_p\left(\frac{\partial p}{\partial V}\right)_T,$$

or

$$\left(\frac{\partial p}{\partial T}\right)_v = ae_T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16\cdot982)$$

since, by definition, the isothermal elasticity, e_T, is

$$e_T = -V\left(\frac{\partial p}{\partial V}\right)_T.$$

Now substituting the expressions (16·981) and (16·982) in (16·98) we get

$$c_p - c_v = Te_Ta^2V \quad . \quad . \quad . \quad . \quad . \quad (16\cdot983)$$

where V is the volume of a gram of the material. It will be observed that the product on the right (if different from zero) is essentially positive. Hence $c_p - c_v$ is always **positive or zero**.

We derive the formulae which express adiabatic relationships by making c_x zero in (16·95), or in (16·972). The condition x is now simply the condition $\phi = $ constant, consequently

$$c_v = -T\left(\frac{\partial p}{\partial T}\right)_v\left(\frac{\partial V}{\partial T}\right)_\phi. \quad . \quad . \quad . \quad (16\cdot99)$$

and

$$c_p = T\left(\frac{\partial V}{\partial T}\right)_p\left(\frac{\partial p}{\partial T}\right)_\phi \quad . \quad . \quad . \quad (16\cdot991)$$

These equations reduce to

$$c_v = -Tae_T\left(\frac{\partial V}{\partial T}\right)_\phi \quad . \quad . \quad . \quad (16\cdot992)$$

and

$$c_p = TaV\left(\frac{\partial p}{\partial T}\right)_\phi \quad . \quad . \quad . \quad . \quad (16\cdot993)$$

Dividing the latter by the former we get

$$\gamma = \frac{c_p}{c_v} = - V\frac{\left(\frac{\partial p}{\partial V}\right)_\phi}{e_T}$$

or

$$\gamma = \frac{c_p}{c_v} = \frac{e_\phi}{e_T} \quad . \quad . \quad . \quad . \quad (16\cdot994)$$

a result we have already established for the special case of a perfect gas.

§ 17.   APPLICATION TO A V. D. WAALS BODY

By differentiating the equation of state,

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT,$$

we obtain

$$\left(\frac{\partial p}{\partial T}\right)_v = \frac{R}{V - b},$$

and

$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{R}{\left(p + \frac{a}{V^2}\right) - \frac{2a}{V^3}(V - b)},$$

or

$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{R(V - b)}{RT - \frac{2a}{V^3}(V - b)^2}.$$

Consequently, on substituting in (16·98),

$$c_p - c_v = \frac{R^2}{R - \frac{2a(V - b)^2}{V^3} \frac{}{T}},$$

which becomes, if we neglect small quantities of the second order,

$$c_p - c_v = \frac{R}{1 - \frac{2a}{RTV}} = R\left(1 + \frac{2a}{RTV}\right) \quad . \quad . \quad (17)$$

For an adiabatic expansion of a v. d. Waals body we find from (16·99)

$$c_v = - \frac{TR}{(V - b)}\left(\frac{\partial V}{\partial T}\right),$$

or dropping the subscript, ϕ,

$$0 = c_v dT + RT\frac{dV}{V - b}.$$

Therefore
$$0 = c_v \frac{dT}{T} + R \frac{dV}{V-b},$$

consequently $0 \equiv c_v d \log T + R d \log (V-b),$

and, if we may take c_v to be a constant we find, on integration,
$$T^{c_v}(V-b)^R = \text{constant},$$

or $T(V-b)^{\frac{R}{c_v}} = \text{constant}$ . . . (17·01)

If we divide (16·94) by T, and substitute for $\left(\frac{\partial p}{\partial T}\right)_v$ the expression appropriate for a v. d. Waals body, we find
$$\frac{dQ}{T} = c_v \frac{dT}{T} + R \frac{dV}{V-b}.$$

Therefore $d\phi = c_v \frac{dT}{T} + R \frac{dV}{V-b},$

and $\phi = c_v \log \frac{T}{T_0} + R \log \frac{(V-b)}{(V_0 - b)}$ . . . (17·02)

where T_0 and V_0 are the temperature and volume at which we have agreed the entropy shall be zero.

§ 17·1. Thermodynamic Potentials

There is a number of functions which are prominent in the application of thermodynamical principles to special problems, and which on account of their properties are called **thermodynamic potentials.** Consider, for instance, any reversible process taking place at constant temperature and pressure. We have
$$dQ = dU + pdV,$$
and therefore $Td\phi = dU + pdV.$
If now T and p are constant during the process,
$$d(T\phi) = dU + d(pV),$$
and consequently
$$0 = d\{U - T\phi + pV\}. \quad . \quad . \quad (17·1)$$
In such reversible processes therefore the function $U - T\phi + pV$ remains constant. This function, which we shall represent by the letter f, is commonly called **the thermodynamic potential.** Its increment df can be written
$$df = (dU - Td\phi + pdV) \\ - \phi dT + Vdp,$$
or since the terms in brackets are collectively zero,
$$df = -\phi dT + Vdp \quad . \quad . \quad . \quad . \quad (17·11)$$
This is obviously a perfect differential, because the differentials of the terms which make up f are themselves perfect differentials.

It follows therefore that

$$- \phi = \left(\frac{\partial f}{\partial T}\right)_p,$$

and

$$V = \left(\frac{\partial f}{\partial p}\right)_T \quad . \quad . \quad . \quad . \quad (17 \cdot 12)$$

It is to this property that the function f owes the name 'potential.'

A similar function is the **free energy** of a system, which we shall represent by the letter F. We arrive at it naturally in inquiring about the external work done by a system during a reversible process taking place at constant temperature. Starting out from

$$T d\phi = dU + p dV,$$

we have

$$p dV = - dU + T d\phi,$$

or, when the temperature is constant,

$$p dV = - d\{U - T\phi\}$$
$$= - dF \quad . \quad . \quad . \quad . \quad . \quad (17 \cdot 13)$$

Or the external work done during such a process is done at the expense of the quantity

$$F = U - T\phi \quad . \quad . \quad . \quad . \quad (17 \cdot 14)$$

The increment dF of F is

$$dF = (dU - T d\phi + p dV) - \phi dT - p dV,$$

and as the expression in brackets is zero,

$$dF = - \phi dT - p dV.$$

This is also a perfect differential, and consequently

$$- \phi = \left(\frac{\partial F}{\partial T}\right)_v,$$

$$- p = \left(\frac{\partial F}{\partial V}\right)_T \quad . \quad . \quad . \quad (17 \cdot 141)$$

If we substitute for ϕ in (17·14) the equivalent expression in (17·141) we have

$$F = U + T\left(\frac{\partial F}{\partial T}\right)_v \quad . \quad . \quad . \quad (17 \cdot 15)$$

This result is known as the Gibbs-Helmholtz formula. If instead of the variables p and V, the equation of state contains other corresponding variables, y and x, (17·15) becomes

$$F = U + T\left(\frac{\partial F}{\partial T}\right)_x \quad . \quad . \quad . \quad (17 \cdot 151)$$

The Gibbs-Helmholtz formula finds its chief applications in cases where y (or p) is constant at constant temperature, i.e. is independent of x (or V).

If we write the equation
$$Td\phi = dU + pdV$$
in the form $$dU = Td\phi - pdV \quad \ldots \ldots (17\text{·}16)$$
we see that U is a function which has similarities with f and F ;

and also that $$T = \left(\frac{\partial U}{\partial \phi}\right)_v,$$

$$-p = \left(\frac{\partial U}{\partial V}\right)_\phi \quad \ldots \ldots (17\text{·}161)$$

The increment, dU, of the internal energy of a system is equal to the quantity of heat, dQ, which would have to be communicated to it, at constant volume, to produce the increment dU. This consideration suggests still another function, namely one which has the property that its increment is equal to the quantity of heat supplied (reversibly) to the system under the condition of constant pressure. Now since
$$dQ = dU + pdV,$$
this condition leads to
$$dQ = d(U + pV).$$
If therefore we represent the function we are inquiring about by G, we have $$G = U + pV. \quad \ldots \ldots (17\text{·}17)$$
It is called **enthalpy** ($\theta \acute{a} \lambda \pi \omega \varsigma \equiv$ warmth, heat). We find for dG,
$$dG = (dU - Td\phi + pdV) + Td\phi + Vdp,$$
or $$dG = Td\phi + Vdp. \quad \ldots \ldots \ldots (17\text{·}171)$$

Consequently $$T = \left(\frac{\partial G}{\partial \phi}\right)_p,$$

$$V = \left(\frac{\partial G}{\partial p}\right)_\phi \quad \ldots \ldots (17\text{·}172)$$

§ 17·2. Maxwell's Thermodynamic Relations

By applying the rule **(16·91)** to each of the differentials df, dG, dF and dU we obtain the four equations

$$-\left(\frac{\partial \phi}{\partial p}\right)_T = \left(\frac{\partial V}{\partial T}\right)_p, \quad (a)$$

$$\left(\frac{\partial T}{\partial p}\right)_\phi = \left(\frac{\partial V}{\partial \phi}\right)_p, \quad (\beta)$$

$$\left(\frac{\partial \phi}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_v, \quad (\gamma)$$

$$\left(\frac{\partial T}{\partial V}\right)_\phi = -\left(\frac{\partial p}{\partial \phi}\right)_v, \quad (\delta) \quad \ldots \ldots (17\text{·}2)$$

known as **Maxwell's thermodynamic relations.** They are given in the order in which Maxwell himself gave them.[1]

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[1] J. Clerk Maxwell : *Theory of Heat.*

§ 17·3. The Experiments of Joule and Kelvin and the
        Realization of the Work Scale of Temperature

In Joule's historic experiments on the expansion of gases
into a previously exhausted region (§ 15·7) the temperature
changes could not be determined even approximately on account
of the relatively enormous heat capacity of the surrounding
vessel and medium, as compared with that of the gas itself.
The experiments only sufficed to show that such temperature
changes were relatively small.   Kelvin devised an experimental
method which evaded the difficulties of the earlier experiments,
and which he, in collaboration with Joule,[1] successfully applied
to a number of gases.   The gas under experiment was forced
by pressure through a porous plug of cotton wool which occupied
a short length (between 2 and 3 inches) of a long tube.   This
latter was immersed in water maintained at a constant tem-
perature.   The part of the tube containing the porous plug was
made of box-wood, $1\frac{1}{2}$ inch in internal diameter.   The box-wood
being a bad conductor, and the temperature gradients small,
no appreciable transfer of heat occurred between the expanding
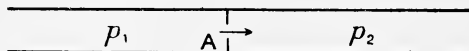
FIG. 17·3

gas and the surrounding medium.   A sensitive thermometer,
placed immediately behind the porous plug, gave the temperature
of the gas on emerging from the plug.   Only the box-wood
part of the tube was thermally insulating, so that before expansion
the gas had the temperature of the surrounding water.   The
pressures on both sides of the plug were maintained constant,
on one side atmospheric pressure and various pressures up to
several atmospheres on the other.   We may visualize the porous
plug as a diaphragm, A (Fig. 17·3), with a minute aperture in
it.   The gas expands through the aperture from a region of
constant high pressure, p_1, into a region of constant low pressure,
p_2.   If V_1 be the volume of the unit mass of the gas at the
pressure, p_1, and V_2 its volume at the lower pressure, p_2, the
external work done *on* the unit mass of the gas will be $p_1 V_1$,
and that done *by* it, $p_2 V_2$.   Consequently the net amount of
work done by the unit mass of the gas in expanding will be

$$p_2 V_2 - p_1 V_1,$$

or $\triangle pV$.   If therefore the gas does not deviate appreciably

[1] Joule : *Scientific Papers*, Vol. II, p. 217.

from Boyle's law over the range of pressures to which it is subjected in the experiments, the net external work done by it (or on it) is zero ; in which case the experiment is essentially Joule's original experiment in a slightly different form. In general however there will be small, but appreciable, deviations from Boyle's law, so that the temperature change accompanying the expansion of the gas in passing through the porous plug will be partly due to this. As there is no appreciable transfer of heat between the gas and the surrounding medium we have

$$\triangle(U + pV) = 0$$

or
$$\triangle G = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17{\cdot}3)$$

The corresponding condition in the original Joule experiment, assuming no transfer of heat, is

$$\triangle U = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17{\cdot}31)$$

The successive steps in the application of thermodynamical principles to the two experiments are given in parallel columns below. Most of them are fairly obvious and are therefore given without detailed explanation :

Joule-Kelvin.

$$\triangle G = 0,$$
$$T \triangle \phi + V \triangle p = 0,$$

$$T\left\{ \left(\frac{\partial \phi}{\partial T}\right)_p \triangle T + \left(\frac{\partial \phi}{\partial p}\right)_T \triangle p \right\}$$
$$+ V \triangle p = 0,$$

$$T\left(\frac{\partial \phi}{\partial T}\right)_p \triangle T$$
$$+ \left\{ V + T\left(\frac{\partial \phi}{\partial p}\right)_T \right\} \triangle p = 0,$$

$$c_p \triangle T + \left\{ V + T\left(\frac{\partial \phi}{\partial p}\right)_T \right\} \triangle p = 0,$$

Applying Maxwell's relation (a) (17·2)

$$c_p \triangle T + \left\{ V - T\left(\frac{\partial V}{\partial T}\right)_p \right\} \triangle p = 0.$$

If $\qquad \xi \equiv \dfrac{\triangle T}{\triangle p},$

$$c_p \xi + \left\{ V - T\left(\frac{\partial V}{\partial T}\right)_p \right\} = 0.$$
$$(17{\cdot}32)$$

Joule.

$$\triangle U = 0,$$
$$T \triangle \phi - p \triangle V = 0.$$

$$T\left\{ \left(\frac{\partial \phi}{\partial T}\right)_v \triangle T + \left(\frac{\partial \phi}{\partial V}\right)_T \triangle V \right\}$$
$$- p \triangle V = 0,$$

$$T\left(\frac{\partial \phi}{\partial T}\right)_v \triangle T$$
$$- \left\{ p - T\left(\frac{\partial \phi}{\partial V}\right)_T \right\} \triangle V = 0,$$

$$c_v \triangle T - \left\{ p - T\left(\frac{\partial \phi}{\partial V}\right)_T \right\} \triangle V = 0,$$

Applying Maxwell's relation (γ) (17·2)

$$c_v \triangle T - \left\{ p - T\left(\frac{\partial p}{\partial T}\right)_v \right\} \triangle V = 0.$$

If $\qquad \eta \equiv \dfrac{\triangle T}{\triangle V},$

$$c_v \eta - \left\{ p - T\left(\frac{\partial p}{\partial T}\right)_v \right\} = 0.$$
$$(17{\cdot}321)$$

It should be observed that, in these two formulae, the temperature is expressed in terms of the work scale ; while in the experiments themselves the temperatures or temperature differences were determined by mercury in glass thermometers. Imagine the temperatures, in the Joule-Kelvin experiments, to be expressed in terms of the constant pressure gas scale of the gas under experiment. This is practicable, since it is only necessary to compare the thermometer actually used with the gas thermometer. Let T' be the temperature on the constant pressure gas scale as defined by

$$pV = RT',$$

p being constant, and R chosen as explained in § 15·1. We have consequently

$$p\left(\frac{\partial V}{\partial T'}\right)_p = R,$$

or

$$\left(\frac{\partial V}{\partial T'}\right)_p = \frac{V}{T'} \quad \cdots \cdots \quad (17\cdot33)$$

The specific heat, c_p, and the ratio, ξ, when expressed in terms of the scale T', may be represented by c'_p and ξ' respectively. It is clear that

$$c_p\,\xi = c_p{'}\,\xi',$$

because specific heat is a quantity with the temperature as denominator, whereas it constitutes the numerator in the quantity ξ, and in the product of the two the peculiarities of the scale of temperature actually used cancel out. We may now re-write (**17·32**), and *mutatis mutandis* (**17·321**), in the manner shown below, and obtain results which enable us to use the observations in the Joule-Kelvin and the Joule experiment to correct the readings of the constant pressure and constant volume gas thermometer (containing the gas experimented on) respectively, so as to get temperatures on the work scale.

Joule-Kelvin.	*Joule.*

$$c_p{'}\xi' + \left\{ V - T\left(\frac{\partial V}{\partial T'}\right)_p \frac{dT'}{dT} \right\} = 0, \qquad c_v{'}\eta' - \left\{ p - T\left(\frac{\partial p}{\partial T'}\right)_v \frac{dT'}{dT} \right\} = 0,$$

Applying (**17·33**)   Applying the formula analogous to (**17·33**)

$$c'_p\xi' + \left\{ V - T\frac{V}{T'}\frac{dT'}{dT} \right\} = 0, \qquad c_v{'}\eta' - \left\{ p - T\frac{p}{T'}\frac{dT'}{dT} \right\} = 0,$$

$$\frac{dT}{T} = \frac{dT'}{T'\left\{ 1 + \dfrac{c'_p\xi'}{V} \right\}} \quad (17\cdot34) \qquad \frac{dT}{T} = \frac{dT'}{T'\left\{ 1 - \dfrac{c_v{'}\eta'}{p} \right\}} \quad (17\cdot341)$$

Taking the left-hand formula (17·34) and integrating over the range of temperature from that of melting ice to that of saturated water vapour at normal pressure, we have

$$\log \frac{T_0 + 100}{T_0} = \int_{T_0'}^{T_0'+100} \frac{dT'}{T'\left\{1 + \frac{c_p'\xi'}{V}\right\}},$$

where T_0 and T_0' are the temperatures of melting ice on Kelvin's scale and on the constant pressure gas scale respectively. The integral on the right is made up of observable quantities only, and can be evaluated. Calling it τ_0, we have

$$T_0 = \frac{100}{e^{\tau_0} - 1} \quad . \quad . \quad . \quad . \quad . \quad (17·35)$$

Similarly for any other temperature, T', on the constant pressure gas scale and the corresponding temperature, T, on Kelvin's work scale, we have

$$\log \frac{T}{T_0} = \int_{T_0'}^{T'} \frac{dT'}{T'\left\{1 + \frac{c_p'\xi'}{V}\right\}}.$$

Calling the integral on the right τ, we have

$$T = \frac{100e^{\tau}}{e^{\tau_0} - 1} \quad . \quad . \quad . \quad . \quad . \quad (17·36)$$

This formula will also serve for the constant volume thermometer if for τ and τ_0 we substitute the values of the corresponding integrals obtained from (17·341). Such an application would, however, have no practical value if we had to rely on estimates of η' derived from experiments of the original Joule type.

The theory of the Joule-Kelvin experiment (and of the Joule experiment) applies not only to gases, but to any sort of fluid up to equations (17·32) and (17·321). Joule and Kelvin found a cooling effect (ξ positive) for all the gases they experimented on, except hydrogen, for which they observed *a very small rise* in temperature (ξ negative). They found the change in temperature to be proportional to the drop in pressure, $p_1 - p_2$, and inversely proportional to the square of the absolute temperature. The cooling effect is of course the basis of the methods of liquefying air which are most extensively used at the present time. Gases like hydrogen and helium, which in the ordinary way exhibit a heating effect, are found at sufficiently low temperatures to be cooled. There is therefore a temperature of

inversion at which ξ changes sign, i.e. becomes zero. From equation (**17·32**) we learn that, when $\xi = 0$,

$$\frac{V}{T} = \left(\frac{\partial V}{\partial T}\right)_p \qquad \ldots \quad \ldots \quad (\textbf{17·37})$$

If we represent the equation of state of the gas by

$$f(T, V, p) = 0 \quad \ldots \quad \ldots \quad (\textbf{17·38})$$

differentiate it with respect to T, keeping p constant, and then equate the expression thus found for $\left(\dfrac{\partial V}{\partial T}\right)$, to $\dfrac{V}{T}$, (**17·37**), we get an equation

$$\phi(T, V, p) = 0 \quad \ldots \quad \ldots \quad (\textbf{17·39})$$

connecting T, V and p, which is true for all states of the gas for which $\xi = 0$. We may eliminate one of the variables, p for example, from (**17·39**), by using the equation of state (**17·38**), and we thus obtain an equation which gives us the temperature of inversion in terms of the volume. Its graphical representation on a TV diagram is called the **curve of inversion.**

For a v. d. Waals body

$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{R}{\left(p + \dfrac{a}{V^2}\right) - \dfrac{2a}{V^3}(V - b)},$$

and therefore by (**17·37**) we have

$$\frac{V}{T_i} = \frac{R}{\left(p + \dfrac{a}{V^2}\right) - \dfrac{2a}{V^3}(V - b)},$$

where T_i is the temperature of inversion. On eliminating p we easily find

$$T_i = \frac{2a}{Rb}\left(\frac{V - b}{V}\right)^2 \quad \ldots \quad \ldots \quad (\textbf{17·391})$$

The following *null method* of realizing the Kelvin work scale is of interest, though it may not be of practical importance. Imagine we have found empirically the equation of state of a gas,

$$f(T'Vp) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (\textbf{17·392})$$

where T' is the temperature in terms of the constant pressure scale of the gas in question, and likewise the equation of the curve of inversion,

$$g(T'V) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (\textbf{17·393})$$

Differentiating (**17·392**) with respect to T', keeping p constant, we obtain an equation

$$\left(\frac{\partial V}{\partial T'}\right)_p = f_1(T', V, p).$$

Multiplying both sides of this by $\dfrac{dT'}{dT}$ we find for states of the gas represented by points on the curve of inversion, by (**17·37**),

$$\frac{V}{T} = f_1(T', V, p)\frac{dT'}{dT}.$$

We can eliminate p and V, by means of (17·392) and (17·393) and thus obtain

$$\frac{dT}{T} = f_2(T')dT' \quad \ldots \quad \ldots \quad (17\text{·}394)$$

which, since the function f_2 is known, enables us to find T in terms of T'.

§ 17·4.   HETEROGENEOUS SYSTEMS

We now turn to systems in which two or more states of aggregation, or **phases,** are in equilibrium with one another. The simplest example is that of a liquid in equilibrium with its vapour. For the range of temperatures below the critical temperature of the substance there exists for each temperature a definite pressure, usually called the **saturation pressure** of the vapour, but more appropriately called the **equilibrium pressure,** under which the liquid and its vapour are in equilibrium. This is represented by the horizontal lines such as BF in Fig. 12·51. According to the theory of v. d. Waals the isothermals have the shape illustrated by ACDEG, assuming the whole of the substance to be in *one* state of aggregation at any given pressure or volume. This is supported by the fact that the portions BC and EF can be experimentally realized. The question arises : What is the situation of the horizontal line BF relatively to the curved line ACDEG ? During the reversible passage of the substance from the state A to the state G, there is a definite increase in its entropy, determined, as we have seen, solely by the positions of the points A and G on the diagram. If therefore the passage occurs isothermally the quantity of heat communicated to the substance will be just the same for either of the alternative paths ABFG or ACEG. On the other hand the increase in the internal energy is also the same for both paths, since this too is determined solely by the positions of the points A and G on the diagram. Thus it follows, by the first law of thermodynamics, that the work done, $\int_A^G pdv$, is the same in both cases, and this means that the area BCD is equal to the area DEF.

§ 17·5.   THE TRIPLE POINT

When we plot the pressures associated with each of the horizontal lines BF (Fig. 12·51) against the corresponding temperatures we get such a curve as OA in Fig. 15·31. For a given point on such a curve, the function f (§ 17·1) has the same value

for a gram of the liquid as for a gram of the vapour, since the conversion from one state to the other takes place reversibly at constant pressure and temperature (**17·1**).   Therefore

$$f_v = f_l \qquad \cdots \qquad (17\text{·}5)$$

and since f_v and f_l are definite functions of the independent variables p and T, equation (**17·5**) is the equation of the curve OA.   Similar remarks apply to the equilibria between liquid and solid and solid and vapour, represented by the curves OB and OC.   The equations of the three curves are therefore

$$\begin{aligned} f_v &= f_l, \\ f_l &= f_s, \\ f_s &= f_v \qquad \cdots \qquad (17\text{·}51) \end{aligned}$$

The point of intersection of OA and OB, being common to both curves, satisfies both of the first two of these equations, and hence for this point

$$f_s = f_v,$$

which shows that it is a point on OC.   In other words the three curves intersect in one point, as the figure has anticipated.   This is called the **triple point.**

§ 17·6.   LATENT HEAT EQUATIONS

Consider two neighbouring points on OA (Fig. 15·31).   By (**17·5**)

$$f_v = f_l,$$

and $\qquad\qquad\qquad f_v + df_v = f_l + df_l,$

and therefore $\qquad\qquad df_v = df_l.$

Consequently by (**17·11**)

$$- \phi_v dT + V_v dp = - \phi_l dT + V_l dp,$$

or $\qquad\qquad (\phi_v - \phi_l)dT = (V_v - V_l)dp.$

Now $\qquad\qquad \phi_v - \phi_l = L/T,$ by (**16·7**),

where L is the latent heat of evaporation.   Therefore

$$L = (V_v - V_l)T\frac{dp}{dT} \qquad \cdots \qquad (17\text{·}6)$$

This is known as **Clapeyron's equation.**

It is important to remember that this formula implies the use of absolute units.   For example p means force per unit area, force being measured by rate of change of momentum ; work is measured by the product of force and distance and L is measured in work units of energy reckoned per gram of the substance.   It is of course immaterial what are the precise fundamental units which have been adopted, whether pound,

foot, Fahrenheit degree, etc., or gram, centimetre, centigrade degree, etc.

As an illustration consider the equilibrium between ice (solid) and water (liquid). The latent heat of fusion is approximately

$$80 \times 4 \cdot 2 \times 10^7 \text{ ergs per gram ;}$$
$$V_l - V_s = -0 \cdot 09 \text{ c.c. per gram.}$$

At normal atmospheric pressure, i.e. 1,014,000 dynes per cm.[2], the equilibrium temperature (so-called melting point) is 273 on Kelvin's scale. Therefore

$$80 \times 4 \cdot 2 \times 10^7 = -0 \cdot 09 \times 273 \times \frac{dp}{dT},$$

where dT is the elevation of the melting point of ice due to the elevation, dp, of the pressure. Hence

$$\frac{dT}{dp} = -7 \cdot 3 \times 10^{-9} \quad \text{approximately,}$$

or the melting point is *lowered* by 0·0073° per each atmosphere increase in pressure.

The Gibbs-Helmholtz formula (**17·15**) provides an alternative way of deriving Clapeyron's equation. For a gram of the vapour and liquid respectively,

$$F_v = U_v + T\left(\frac{\partial F_v}{\partial T}\right)_V,$$

$$F_l = U_l + T\left(\frac{\partial F_l}{\partial T}\right)_V.$$

Hence
$$F_v - F_l = U_v - U_l + T\left(\frac{\partial(F_v - F_l)}{\partial T}\right)_V.$$

Now in this case
$$F_v - F_l = -p(V_v - V_l),$$

therefore
$$-p(V_v - V_l) = U_v - U_l - (V_v - V_l)T\frac{dp}{dT},$$

or
$$U_v - U_l + p(V_v - V_l) = (V_v - V_l)T\frac{dp}{dT}.$$

This is Clapeyron's equation, since the left-hand member is obviously identical with the latent heat. Finally it will be noted that this equation is a special case of the more general formula (**16·94**).

Let us now turn to the variation of the latent heat with temperature. It is convenient to make use of the constant pressure lines of the substance (e.g. water and water vapour) on a T, ϕ diagram (Fig. 17·6). Starting at a point A, imagine heat to be communicated to the liquid reversibly at constant pressure. The entropy and temperature will both increase until a point B is reached for which the temperature is the equilibrium temperature of the liquid and its vapour for the particular pressure chosen. The reversible communication of heat is now associated with reversible vaporization, the temperature remaining

constant till the whole of the liquid is vaporized. This stage is represented by the horizontal line BC. Beyond the point, C, the curve will again ascend, as shown by CD. For a slightly higher pressure we have a corresponding curve A'B'C'D'. We may represent the equilibrium temperatures and latent heats corresponding to BC and B'C' by T, $T + dT$ and L, $L + dL$ respectively. The broken line BB' represents, for different temperatures, states of the liquid in which it is in equilibrium with its vapour. Similarly CC' represents saturated vapour at different temperatures. Consider now the heat communicated to the substance when it is taken round the cycle BB'C'CB. Let c_1 be the specific heat of the liquid when in equilibrium with its vapour, and c_2 that of the saturated vapour. The net amount of heat communicated during the cycle is obviously
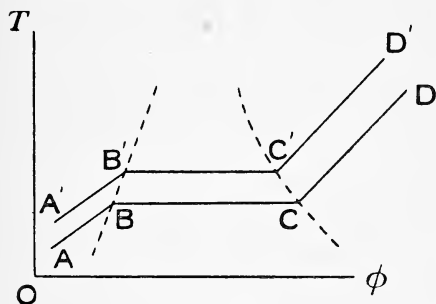
$$c_1 dT + L + dL - c_2 dT - L$$

or

$$(c_1 - c_2)dT + dL \quad . \quad . \quad . \quad . \quad (17\cdot61)$$

But we have already seen (§ 16·6) that this is equal to the area of the closed loop, i.e. to

$$dT \times (BC),$$

or

$$dT(\phi_v - \phi_l),$$

or, finally,

$$\frac{L dT}{T} \quad . \quad . \quad . \quad . \quad . \quad (17\cdot62)$$

On equating (17·61) and (17·62) we get

$$c_2 - c_1 = \frac{dL}{dT} - \frac{L}{T} \quad . \quad . \quad . \quad (17\cdot63)$$

§ 17·7. THE PHASE RULE

Turning again to the equilibrium between two phases of a single constituent, e.g. water, we have seen that we can represent it by a curve on a pT diagram ; for liquid and vapour the curve OA (Fig. 15·31). Within the limits between which these phases can exist we may have equilibrium at any temperature we like to choose ; but having once fixed the temperature there is only one pressure under which equilibrium is possible. Or on the

FIG. 17·6

other hand we may choose any pressure we like, but there will then be only one temperature at which equilibrium is possible. We say the system of liquid and vapour has one **degree of freedom.** The equilibrium of *three* simultaneous phases is represented by a *single point*, O, on the diagram. In this case there are no degrees of freedom at all. In the case of only one phase, e.g. liquid, there are obviously two degrees of freedom. Over the range of pressures and temperatures for which this phase can exist we may choose both arbitrarily and independently of one another. These facts are instances of a simple general rule due to Willard-Gibbs, and known as the **Phase Rule.** It may be stated in the following form :

$$F + P = C + 2 \quad . \quad . \quad . \quad . \quad (17\cdot7)$$

where F is the number of degrees of freedom when P phases are in equilibrium, the number of constituents being C. As an illustration of the case of two constituents, let us take water and a soluble salt. Consider the phases, ice, solid salt, solution of salt in water, and water vapour. For two phases, e.g. solution and vapour, the rule gives,

$$F + 2 = 2 + 2,$$

or two degrees of freedom. This means that we may, for example, choose both pressure and temperature (within the limits between which these phases can exist) at will. Equilibrium will be always possible at some definite concentration of the solution, or we may adjust at will the concentration and temperature ; there will then be a definite pressure under which the two phases are in equilibrium. When three phases are in equilibrium, for example ice, solution and vapour, there is only one degree of freedom.

We can establish the phase rule in the following way : In any reversible transference of one or more constituents from one phase to another (i.e. transference under equilibrium conditions) the function f for the whole system remains unaltered, if we keep the pressure and temperature constant (§ 17·1). Therefore

$$\delta f = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17\cdot71)$$

If there are P phases, f is a sum of contributions from each phase, or

$$f = f' + f'' + f''' + \ldots + f^{(P)} \quad . \quad . \quad . \quad (17\cdot72)$$

and consequently

$$\delta f = \delta f' + \delta f'' + \delta f''' + \ldots + \delta f^{(P)} = 0 \quad . \quad . \quad (17\cdot721)$$

In any redistribution of the constituents among the P phases, let

$$\delta m_1', \ \delta m_1'', \ \delta m_1''', \ \ldots \ \delta m_1^{(P)}$$

represent the increments of constituent number 1 in the P phases respectively and

$$\delta m_2', \ \delta m_2'', \ \delta m_2''', \ \ldots \ \delta m_2^{(P)}$$

those of constituent number 2 in the P phases respectively, and so on.

Since the total masses m_1, m_2, . . . m_c, are given,

$$\delta m_1' + \delta m_1'' + \delta m_1''' + \ldots + \delta m_1^{(P)} = 0,$$
$$\delta m_2' + \delta m_2'' + \delta m_2''' + \ldots + \delta m_2^{(P)} = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\delta m_o' + \delta m_o'' + \delta m_o''' + \ldots + \delta m_o^{(P)} = 0 \quad . \quad \textbf{(17·73)}$$

Taking any one of these equations, the first one for instance, we may choose only $P - 1$ of the δm_1's arbitrarily, the remaining one being determined by the equation. So that altogether there are $C(P - 1)$ δm's only which we may choose arbitrarily. Let us represent them by

$$\delta x_1, \; \delta x_2, \; \delta x_3, \; \ldots \; \delta x_{C(P-1)}.$$

The condition for equilibrium (**17·71**) or (**17·721**) now becomes

$$\frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f}{\partial x_{C(P-1)}}\delta x_{C(P-1)} = 0, \quad . \quad \textbf{(17·74)}$$

and since the δx's are arbitrary, we have

$$\frac{\partial f}{\partial x_1} = 0,$$

$$\frac{\partial f}{\partial x_2} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{\partial f}{\partial x_{C(P-1)}} = 0 \quad . \quad . \quad . \quad . \quad . \quad \textbf{(17·75)}$$

These $C(P - 1)$ conditions are necessary and sufficient for the equilibrium of the P phases at some given pressure and temperature.

Let us now consider how many data are required to fix the state of the system. To begin with we have the two data pressure and temperature. In addition to these we require the data fixing the constitution of each phase. For each phase $C - 1$ data are evidently necessary for given total masses of the C constituents, since the character of a phase is determined by the $C - 1$ ratios

$$\frac{m_2'}{m_1'}, \; \frac{m_3'}{m_1'}, \; \ldots \; \frac{m_o'}{m_1'}$$

of the masses of the C constituents present in it. The constitution of the P phases is therefore determined by $P(C - 1)$ data. Adding to these the 2 data, pressure and temperature, mentioned above, we require altogether

$$2 + P(C - 1)$$

data to completely describe the state of the system. We have already seen that $C(P - 1)$ relations must exist between them, and there remain over consequently at our arbitrary disposal

$$2 + P(C - 1) - C(P - 1)$$
or $\quad\quad 2 - P + C$ factors.

This means that we may choose $2 - P + C$ of the independent variables, which determine the state of the system, quite arbitrarily and still have P phases in equilibrium, i.e.

$$F = 2 - P + C.$$

§ 17·8.   DILUTE SOLUTIONS

A solution of a crystalline or other body, in water for example, has a lower equilibrium vapour pressure than the pure solvent.   We can explain this in the following way :   Imagine two vessels A and B in an enclosure (Fig. 17·8), the former containing the pure solvent, the latter a dilute solution, and the rest of the enclosure only the vapour of the solvent.   If the two levels a and b were initially coincident, the surface of the solution would function as a semi-permeable membrane, and vapour would condense into B until finally a difference in level, h, equivalent to the osmotic pressure, P, of the solution in B, became established (§ 12·4).   When this equilibrium condition exists, the vapour pressure,
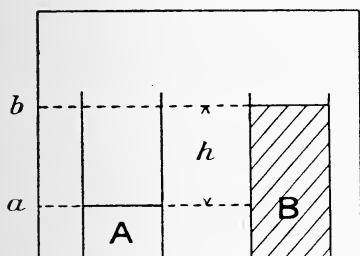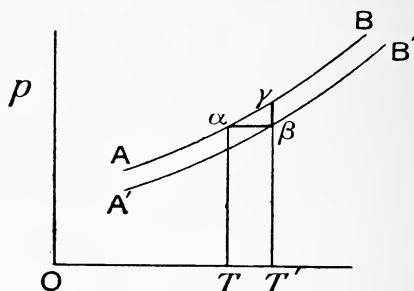
FIG. 17·8   FIG. 17·81

p', must be the same at all points in the horizontal plane, b, and if p be the pressure at the lower level, a, i.e. the equilibrium pressure between the vapour and the pure solvent, obviously

$$p - p' = \delta g h,$$

where δ is the vapour density, which we may take to be approximately uniform.   On the other hand the osmotic pressure, P, is expressed by

$$P = \varrho g h,$$

where ϱ is the density of the solution, or, in the case of a dilute solution, the density of the solvent itself.   Hence

$$\frac{p - p'}{P} = \frac{\delta}{\varrho} \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17\text{·}8)$$

The equilibrium between the vapour and the pure solvent, and that between the vapour and the dilute solution, are represented by the curves AB and A′B′ respectively in Fig. 17·81.   Let T be the boiling point of the pure solvent, i.e. the equilibrium temperature for the solvent and its vapour, when the pressure is the normal pressure of 76 cm. of mercury. The boiling point of the solution will be T′, a little higher as the diagram explains.

Now by Clapeyron's equation (17·6)

$$L = VT\frac{(\beta\gamma)}{(\alpha\beta)}$$

where L is the latent heat of the solvent, V is the volume of the unit mass of the vapour (we have neglected the volume of the unit mass of the

liquid, since it is small by comparison). This equation may be written
in the form

$$L = \frac{1}{\delta}T\frac{p - p'}{T' - T},$$

since $V\delta = 1$. Combining this with equation (17·8), we get

$$T' - T = \frac{TP}{\varrho L} \quad \text{. . . . . . . (17·81)}$$

for the excess of the boiling point of a dilute solution over that of the
pure solvent, and there is obviously an analogous formula for the excess
of the equilibrium temperature of solution and solid solvent over that
of liquid and solid solvent.

The kinetic theory suggests (§ 12·8) that in a dilute solution the
relation between osmotic pressure, volume of solution and temperature
is identical with the perfect gas equation, to a first approximation at
any rate. Therefore

$$PV = RT \quad \text{. . . . . . . (17·82)}$$

where P is the osmotic pressure, V the volume of a gram molecule of
the dissolved substance, and R is the gas constant $(8\cdot315 \times 10^7$ ergs
per ° C.) for a gram molecule (we are assuming that the ultimate particles
of the dissolved substance in the solution are molecules, i.e. that it does
not dissociate, nor associate). If σ is the concentration, i.e. the quantity
of dissolved substance per unit volume, and M its molecular weight,

$$V = \frac{M}{\sigma},$$

and therefore

$$\frac{PM}{\sigma} = RT \quad \text{. . . . . . . (17·82)}$$

Combining this with (17·81) we find

$$T' - T = \frac{RT^2}{ML}\frac{\sigma}{\varrho} \quad \text{. . . . . . (17·83)}$$

a formula which enables an approximate estimate to be made of the
molecular weight of a body from the elevation of the boiling point due
to dissolving it in a suitable solvent.

We have assumed that the dissolved body does not dissociate (nor
associate). If each molecule in solution were to break up into two parts
(ions), the osmotic pressure would of course be twice that which would
result if no such dissociation occurred, and conversely if association of
the molecules to form larger particles were to occur, the osmotic pressure
would be correspondingly lower. This is the reason for the abnormally
low osmotic pressures of colloidal solutions. In aqueous solutions of
crystalline bodies, the ultimate particles in solution are always, or in
most cases, either molecules of the dissolved substance, or ions into
which it dissociates. In the case of cane sugar (and other non-electrolytic
crystalline bodies) the osmotic pressure is quite close to that which we
should calculate on the assumption that it occupied, in gaseous form,
a volume equal to that of the (dilute) solution, without dissociating. In
the case of common salt (and similar electrolytic bodies) the osmotic
pressure in dilute solutions is approximately twice that of the equivalent
solution of cane sugar, indicating that each molecule dissociates into
two particles (ions). The phenomena of electrolysis furnish independent
evidence in support of this view. A rough classification of bodies is

usually made into **crystalloids** and **colloids.**   Cane sugar and common salt are examples of the former class.   Their solutions are characterized by high osmotic pressure, which we explain by the supposition that the ultimate particles of the dissolved substance in solution are molecules or still smaller particles into which their molecules have broken up. Colloids on the other hand are substances the aqueous solutions of which have low osmotic pressures, the ultimate particles of such substances, when in solution, ranging from the order of magnitude of molecules at one extreme to Perrin's visible spherules at the other.

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